A New Partial Proof of Pythagoras' Theorem

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Abstract

This paper proposes an elegant and simple proof of Pythagoras' Theorem. The proof starts by rotating the non-hypotenuse shorter side on an arc towards the other non-hypotenuse side, then computing a value, x, which starts as a negative value, but increases as the rotation happens. When that value hits zero, the identity is at hand. However, although the proposed work provides a fresh perspective on Pythagoras' Theorem, it is not complete. Further suggestions to complete the proof are proposed.

Keywords: Pythagoras' Theorem, Proof by Integration, Novel Approach.

1. INTRODUCTION

Pythagoras' Theorem, the most timeless and revered identity in mathematics, stands as a pivotal point where the worlds of geometry and algebra converge, bridging the gap between art and science. It's probably the most widely used theorems in mathematics. From the dawn of civilization, as humans ventured into structured reasoning and scientific inquiry, the Pythagorean Theorem has been a constant companion and as long as mathematics continues to be the language of the universe, this theorem will retain its esteemed status (Puiu).

The theorem, in simple terms, states that in a right-angled triangle, the square of the length of the hypotenuse (the side opposite the right angle) is equal to the sum of the squares of the lengths of the other two sides. It is named after Pythagoras, an ancient Greek mathematician who is credited with its discovery, but evidence suggests that the relationship was known to many civilizations even earlier (Maor 2007).

Over the centuries, the theorem has found varied applications. In architecture and engineering, it has been used to ensure the construction of right angles and accurate measurements. In physics, it underpins the study of vectors and motion. In astronomy, it is used to calculate distances across the cosmos. Even in contemporary times, it is central to algorithms within computer graphics and gaming.

The theorem itself has been proven in many different ways (Powell) over the centuries by a wide array of scholars, from the amateurs to the professionals, and it continues to be an important area of research in mathematics. Its simplicity and elegance make it one of the most beautiful and fascinating identities in all of mathematics.

There are dozens of proofs for Pythagoras' theorem. There is even one attributed to James Garfield, the 20th President of the United States (Kolpas, 2016). This paper seeks to contribute to this rich heritage by proposing a new geometric approach to proving Pythagoras' Theorem. The approach involves an innovative rotational movement and analysis of the changes in certain quantities as the rotation progresses. While this approach is not presented as a complete proof, it offers a fresh perspective and serves as a stepping stone. In essence, it is an invitation to scholars, teachers, students, and math enthusiasts to engage in a collective endeavor to explore new avenues and, perhaps, find an elegant path that leads to a fully-fledged proof.

2. THE SETUP

Consider a right triangle ABC, right angled at A, as shown in Figure 1. The length of the sides are a, b, and c respectively. Assume that b is smaller than a and draw a semi-circle whose center is A and radius is AB. The length of the radius is b. Thesemi-circle intercepts AC at N and the extension of AC at point P. Naturally AP is equal to AN whichin turn is equal to b.

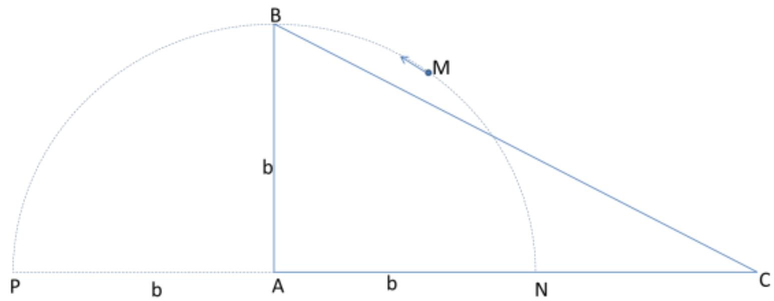

FIGURE 1: New Pythagoras' Proof Setup.

Let M be a point that floats on the semi-circle. It starts at N and moves up towards B and then continues further until it lands on P. Denote m the distance between C and M. Finally, compute the expression x as follows:

$x=m^2-b^2-a^2$

The value of x changes as M slides on the semi-circle from point N towards point B, and eventually arrives at point P.x is dependent on three entities:

m^2 which increase as M moves towards P.
b^2 which is constant no matter where M is.
a^2 which is constant no matter where M is.

It's clear that x increases as M moves on its path from N to P.

3. THE PROPOSAL

Let's compute x when M is at two points of interest, N and P.

When M is at N, m is equal to b-a. Computing x:

$x=(b-a)^2-a^2-b^2$

And simplifying:

$x=-2ba$

When M is at P, m is equal to b+a. Computing x:

$x=(b+a)^2-a^2-b^2$

Simplifying:

$x=2ba$

So,
when M is at N, $x=-2ba$

when M is at P, x=2ba

In essence, the value of x increases from:

-2ba to +2ba as M moves from N to P.

Which stands to reason that when M is half way, at point B, x is equal to zero. At that point m=c. Which means:

$0=c^2-a^2-b^2$

And therefore:

$c^2=a^2+b^2$

That's the new proof. It's Simple, it's elegant and it's straightforward. However it's incomplete!

4. THE GAP

The major assumptionin this argument is the concept that as the point 'M' traverses the arc of the semi-circle from N to P, the value of x escalates from an initial value of -2ab to a terminal value of +2ab. This proposition forms the crux of the logic being employed, but a critical gap remains.

As M ascends the arc from N towards B, and subsequently descends from B towards P, the rate of increase in the value of x is not yet proven to be the same on the ascending journey as it is in the descending one. The symmetry of the geometric setup might suggest that x increases at a consistent rate throughout the movement of M. However, this intuitive proposition lacks formal substantiation. The fundamental challenge here, and the essence of the gap we are referring to in this section, is to prove that x indeed increases by precisely 2ab as M travels from N to B.

To dissect this problem, let us analyze the constituents of x. The expression for x encompasses three terms: m^2, b^2, and a^2. It is essential to recognize that as M shifts its position along the arc from N to B, the variables b and a remain constant. Consequently, any change in x is solely attributable to changes in m^2, the square of the distance between M and C.

Thus, for x to experience an increase of 2ab during the ascent of M from N to B, it is imperative that m^2 correspondingly increases by 2ab. This raises two interlinked questions that are central to addressing the gap: What is the nature of the relationship between m and the position of M along the arc? And how does this relationship govern the changes in x?

The precise quantification of this relationship is the missing piece in our puzzle. A rigorous mathematical treatment is needed to demonstrate whether the symmetry and geometric constructs inherent in the setup are sufficient to warrant the assumption of a uniform rate of increase of x.

This gap in the logic is significant as it holds the key to affirming or refuting the validity of the presented approach. It presents an intriguing mathematical challenge.

We will present in the coming sections a couple of proposals on how to approach solving this gap.

5. APPROACH 1

A conceivable strategy to address the gap identified in the previous section is to evaluate the collective change in m^2 as M traverses the arc of the semi-circle. This can be accomplished by summing up infinitesimal increases in m^2 as 'M' moves through infinitely small steps on the arc.

This section provides a mathematical foundation to this approach and identifies possible pathways for future exploration.

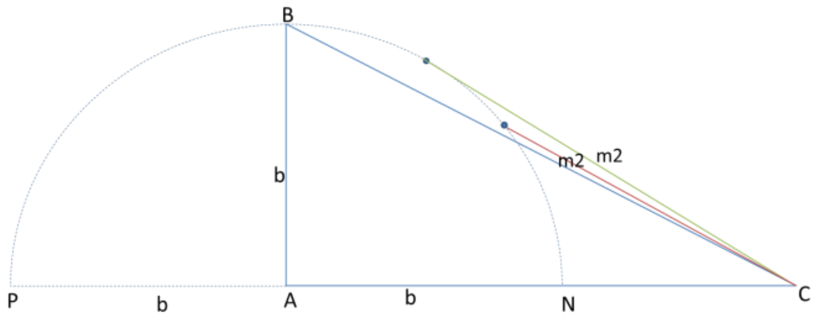

FIGURE 2: Infinitesimal Changes in m^2.

As depicted in Figure 2, let's consider that M moves from an initial position M_1 to a new position M_2 on the arc. This motion is characterized by an infinitesimally small change in the distance from C to M, denoted by Δm:

$\Delta m = m_2 - m_1$

Recall from the previous section that the only component of x that increases as M moves from M_1 to M_2, is m (since both b and a are constants.) Therefore as M moves from M_1 to M_2, the change in x are the same as the changes in m^2which is:

$\Delta m^2 = m_2^2 - m_1^2$

Repeating for every increment starting with M at point N and ending with M at P, the change in $m^2$is:

$\Delta m^2 = \sum(m_2^2 - m_1^2)$ as m moves from: m=b-a tom=b+a

If we make the increments infinitesimally small, the expression becomes:

$\Delta m^2 = \sum (m_2^2 - m_1^2) = \sum (m_2 + m_1)(m_2 - m_1)$

Since M_2 and M_1 are infinitesimally close, we can safely assume that m_2+m_1 is 2m and m_2-m_1 is dm. Hence:

$\Delta x = \int 2m.dm$

As M moves from point N to point P, the length of m moves from m=b-a to m=b+a and the integral above becomes the definite integral:

$\Delta x = \int_{b-a}^{b+a} 2m.\,dm$

Solving we get:

$\Delta x = m^2$, as m changes from b-a tob+a, which computes to 4ab.

This is in agreement with our view that x increases by 4ab as m moves from N to P.

What happens when we make M stop at B. The same integral applies except that the end point is B which means the definite integral becomes:

$\Delta x = \int_{b-a}^{c} 2m.dm$

And $\Delta x = c^2 - b^2 - a^2 + 2ab$

If only we can prove that this value, Δx, is equal to 2ab, we'll have

$c^2 - b^2 - a^2 = 0$, or $c^2 = a^2 + b^2$

And that will prove Pythagoras' Theorem.

So the question finally becomes how to prove that the above integration is equal to 2ba.

A potential solution could be to prove that:

$\Delta m = \int_{b-a}^{b+a} 2m.dm$ is equal to $\int_{0}^{b} 2a.da$

An alternative is to prove:

$\Delta m = \int_{b-a}^{b+a} 2m.dm$ is equal to $\int_{0}^{a} 2b.db$

Both cases will solve the problem. There must be some similar triangle proportionality involving the points C, M_1, M_2, and A that could prove either of the two above identities. Further research is needed.

6. APPROACH 2

A distinct method to establish the relationship between x and the movement of point M involves exploiting the symmetry of the semi-circle. In particular, we can investigate whether the increase in x during the ascent of M from N to B is equal to the increase in x during the descent of M from B to P. If this symmetry holds, then the increase in x during the ascent (N to B) will be half of the total increase of x during the complete movement (N to P).

We established earlier that the total increase in x as M moves from N to P is 4ab. If the increases are symmetric about the highest point B, then the increase in x when M reaches B is 2ab. According to the results of the previous section, this is sufficient to establish Pythagoras' Theorem.

To explore this symmetry, let's consider two points M_1 and M_2 on one side of the semi-circle and two symmetrical points M_3 and M_4 on the other side, as shown in Figure 3. We want to verify that the change in x, denoted by Δx_1, as m increases from m_1 to m_2, is equal to the change in x, denoted by Δx_2, as m increases from m3 to m4.

Mathematically, we can represent Δx_1 as:

$\Delta x_1 = m_2^2 - m_1^2 = (m_2 - m_1)(m_2 + m_1)$

As the change in m becomes infinitesimally small, it's reasonable to assume $m_2$is equal to m_1. Hence, we can simplify Δx_1:

$\Delta x_1 = 2m_1(m_2 - m_1)$

Similarly:

$\Delta x_2 = 2m_3(m_4 - m_3)$

We know that m_3 is bigger than m1. Therefore for Δx_2 to be equal to Δx_1:

m_3/m_1 should be equal to $(m_2-m_1)/(m_4-m_3)$

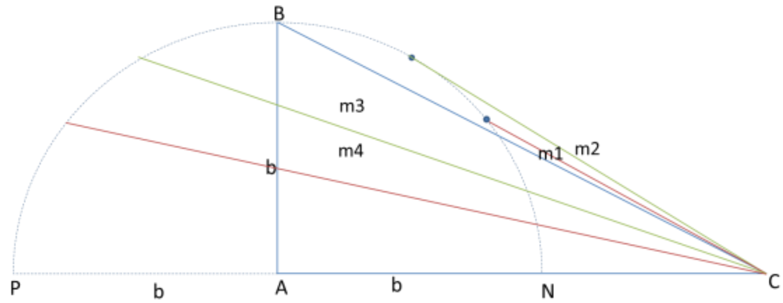

FIGURE 3: Semi-Circle Symmetry.

This relation suggests a proportional symmetry in the value of m on either side of the highest point B.

The symmetry is not an arbitrary assumption but is based on the geometric properties of the semi-circle. A potential avenue of exploration could involve proving that the triangles formed by the points C, M_1, M_2, and A and the corresponding symmetrical triangles formed by C, M_3, M_4 an d A have some proportionality relation. This in turn could lead to proving Δx_1 and Δx_1 are equal.

Moreover, understanding the geometric relationships between these points and the center A of the semi-circle could offer additional insights into the angles involved and their relations to m.

7. CONCLUSION

In this paper, we introduce a novel approach to proving Pythagoras' Theorem, which hinges on the innovative idea of rotating the shorter non-hypotenuse side of a right-angled triangle along an arc towards its counterpart. As the side rotates, a value, x, begins in the negative but gradually increases. Through this exploration of geometric relationships and the movement of a point on a semi-circle, we seek to offer an intuitive grasp of the theorem's essence.

However, the proof is not without its challenges. While the methodology is simple in concept, the execution requires addressing some nuanced geometric relationships and symmetries. In particular, the movement of the point along the semi-circle and the changes in lengths and potentially areas need to be analytically related to the Pythagorean identity.

The study reveals two primary directions that demand further research. The first involves the rigorous evaluation of the integral expressions representing the change in the square of the lengths. The second investigates the geometric symmetry of the semi-circle and its implications on the relationships between the sides.

By encouraging further research and engagement with the material, we hope that this novel approach will either culminate in a new proof of Pythagoras' Theorem or at the very least, contribute to the rich tapestry of geometric insights and understandings that this iconic mathematical principle has inspired through the ages.

8. REFERENCES

Aaboe, Asger. (1997). Episodes from the Early History of Mathematics.Random House.

Eves, Howard. (1981). Great Moments in Mathematics (After 1650).Dolciani Mathematical Expositions.

Euclid. (1956). Elements. T.L. Heath (trans.).

Kolpas, Sid J. (2016). Mathematical Treasure: James A. Garfield's Proof of the Pythagorean Theorem. Mathematical Association of America. https://maa.org/press/periodicals/convergence/mathematical-treasure-james-a-garfields-proof-of-the-pythagorean-theorem

Loomis, Elisha Scott. (1940). The Pythagorean Proposition: Its Demonstrations Analyzed and Classified, and Bibliography of Sources for Data of the Four Kinds of 'Proofs'. National Council of Teachers of Mathematics.

Maor, Eli. (2007). The Pythagorean Theorem: A 4,000-Year History. Princeton University Press.

Nelsen, Roger B. (1993). Proofs without Words: Exercises in Visual Thinking. Mathematical Association of America.

Ostermann, Alexander, &Wanner, Gerhard.(2012). Geometry by Its History.Springer Science & Business Media.

Posamentier, Alfred, & Lehmann, Ingmar. (2012). The Secrets of Triangles. Prometheus Books.

Powell, Anthony. (n.d.).Pythagorean Theorem. https://www.cut-the-knot.org/pythagoras/. Last Access: March 2023.

Puiu, Tibi. (n.d.). 10 Beautiful Visual Mathematical Proofs: Elegance and Simplicity. https://www.zmescience.com/feature-post/natural-sciences/mathematics/10-beautiful-visual-mathematical-proofs-elegance-and-simplicity/. Last Access: March 2023.

Stillwell, John. (2010). Mathematics and its History.Springer Science & Business Media.

Sutevski, Dragan. (n.d.). The Concept of Pythagoras Theorem and Why It is Important? Entrepreneurship in a Box. https://www.entrepreneurshipinabox.com/24522/why-is-pythagoras-theorem-important/. Last Access: March 2023.

Weisstein, Eric W. (n.d.). Pythagorean Theorem. MathWorld--A Wolfram Web Resource. [http://mathworld.wolfram.com/PythagoreanTheorem.html]. Last Access: March 2023.