Reliability Evaluation of Stochastic-Flow Network under Quickest Path and System Capacity Constraints

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Abstract

This paper proposes an algorithm to evaluate the probability that d units of data can be sent from the source node to the sink node through a valid group of paths on a network. The conditions of transmission are such that the transmission time of each path belonging to this valid group of paths does not exceed the specified upper bound T, and that the maximal capacity of that path is not less than the specified lower bound C_s (the required system capacity). Such a probability, which is called the system reliability, is denoted by $R_{(d,Cs,T)}$. Based on minimal paths, the algorithm generates all the lower boundary points for (d,C_s,T), and the system reliability can then be calculated in terms of all the lower boundary points for (d,C_s,T) by applying the inclusion-exclusion rule.

Keywords: Time and capacity constraints, minimal path, stochastic-flow network, system reliability.

1. INTRODUCTION

The system reliability of a flow network R_d is the probability that the maximum flow of the network is not less than a given demand d (which is known as a single commodity). For the case when each arc has several capacities and may fail, [1] presented an algorithm to evaluate R_d in terms of minimal pathsets. In [2], Lin presents an algorithm to evaluate R_d for the case when both the arc and the node have several capacities and may fail. A flow network with two commodities has been studied in [3].

The system reliability of a flow network $R_{(d,C)}$ is defined as the probability that d units of flow can be transmitted from the source node to the sink node, such that the total transmission cost is less than or equal to C; this can be computed in terms of minimal path vectors to level (d,C) (named (d,C)-MPs). In [4], Lin presented an algorithm to generate all (d,C)-MPs of such a system for each level (d,C) in terms of minimal path sets, considering the fact that each arc has several capacities and may fail. For the case when each node and arc having a designated capacity has a different lower level owing to various partial and complete failures, Lin [5] proposed an efficient algorithm, based on minimal paths, that generates all lower boundary points for (d,C). The system reliability can then be calculated in terms of all lower boundary points for (d,C) by applying the inclusion-exclusion rule. The system reliability $R_{(d,C)}$ of a multicommodity flow network has been studied in [6] and [7].

The system reliability of a flow network $R_{(d,T)}$ is defined as the probability that d units of data can be sent from the source to the sink through a stochastic-flow network within T units of time. Based on minimal paths, Lin presented an algorithm to calculate $R_{(d,T)}$ [8].

The idea of considering path capacity and the required system capacity C_s in the reliability evaluation is referred to in [9]. The authors stated that a system is good if and only if it is possible to successfully transmit the required capacity from the source node to the sink node.

In this paper, we will extend the idea of using C_s to a flow network. We suppose that the system has a limited capacity value C_s, and we want to send d units of data within T units of time. Thus, we have a new measure of the system reliability of a flow network denoted by $R_{(d,Cs,T)}$. This new measure is defined as the probability that d units of data can be sent from the source node to the sink node through a valid group of paths on the network within the transmission time (T) under the required system capacity (C_s) constraints.

2. NOTATIONS and ASSUMPTIONS

2.1 Notations

G(A, N, C) A stochastic-flow network with a set of arcs **A =** $\{a_i \mid 1 \le i \le n\}$, a set of nodes N, and $C = \{C_1, C_2, ..., C_n\}$ with C_i (an integer) being the maximum capacity of each arc a_i.

X Capacity vector; X = ($x_1, x_2, \ldots, x_n$).
MPs Minimal paths.
mp_j Minimal path no. j; j = 1, 2,…, m.
l_i The lead time of arc a_i.
C_s The required system capacity.
$R_{(d,Cs,T)}$ The system reliability for a given demand d under the constraints of T and C_s.

2.2 Assumptions

1- The capacity of each component a_i is an integer-valued random variable that takes values $0 < 1 < 2 < ... < M^i$ according to a given distribution.
2- The flow in G must satisfy the so-called flow-conservation law.
3- The capacities of different components are statistically independent.

3. AN ALGORITHM for COMPUTING $R_{(d,Cs,T)}$

3.1 Definition of lower boundary points for (d,C_s,T).
If X is a minimal capacity vector such that the network can send d units of data from the source to the sink within T units of time under a system capacity C_s, then X is a lower boundary point for (d,C_s,T).

3.2 Generate all Lower Boundary Points for (d,C_s,T).

In the following steps, for each minimal path mp_j = {a_{j1}, a_{j2},…, a_{jn}}, we show how to find the minimal capacity vector X^j = ($x_1, x_2, \ldots, x_i, \ldots, x_n$) such that the network sends d units of data within T units of time under a maximum system capacity C_s.

1. For all mp_j, examine the path capacities C_{mpj} as
$$C_{mpj} = \min\{C_i | a_i \in mp_j\},\ j = 1, 2, \ldots, m. \qquad \ldots(1)$$
2. For all mp_j, calculate the transmission time of the path T_j as
$$T_j = \sum_{i=1}^{n} \{l_i \mid a_i \in mp_j\} + \lceil d / C_s \rceil \qquad \ldots(2)$$
3. Determine the valid group paths, $V_{mp} = \{mp_j | \ C_{mpj} \ge C_s \text{ and } T_j \le T,\ j = 1, 2, \ldots, m\}$.
4. Generate the system capacity vector X^j = ($x_1, x_2, \ldots, x_i, \ldots, x_n$) for each mp_j that belongs to V_{mp} as follows:
$$x_i = \begin{cases} C_s & \text{if } a_i \in mp_j \\ 0 & \text{otherwise} \end{cases} \qquad \ldots(3)$$
where x_i is an element of X^j.

Lemma 1. If X is a lower boundary point for (d,C_s,T), then the system capacity under X is greater than or equal to C_s, and the minimum transmission time under X is less than T.

Lemma 2. The set generated by the algorithm 3.2—X^1, X^2,…, X^q—is the set of lower boundary points for (d,C_s,T).

3.3 Evaluate $R_{(d,Cs,T)}$

If X^1, X^2,…, X^q are the collection of all (d,C_s,T)-mp, then the system reliability $R_{d,Cs,T}$ is defined by
$$R_{(d,Cs,T)} = \Pr\{\bigcup_{i=1}^{q} \{Y \mid Y \ge X^i\}\} \qquad \ldots(4)$$
where $\Pr\{Y\} = \Pr\{y_1\} \cdot \Pr\{y_2\} \cdot \ldots \cdot \Pr\{y_n\}$. We will use the inclusion-execlusion rule presented in [10] to calculate $R_{d,Cs,T}$ as follows:

If $A_1 = \{Y \mid Y \geq X^1\},\ A_2 = \{Y \mid Y \geq X^2\},\ \ldots,\ A_q = \{Y \mid Y \geq X^q\}$, then apply the inclusion-exclusion rule to calculate $\mathbf{R_{d,Cs,T}}$ using the following relationship**:**

$$R_{(d,Cs,T)} = \sum_{i} \Pr\{A_i\} - \sum_{i \neq j} \Pr\{A_i \cap A_j\} + \sum_{i \neq j \neq k} \Pr\{A_i \cap A_j \cap A_k\} - \ldots + \\ + (-1)^{q-1} \Pr\{A_1 \cap A_2 \cap \ldots \cap A_q\} \qquad \ldots(5)$$

4. AN ILLUSTRATIVE EXAMPLE

Here, we use the network in Fig. 1 that was studied in [8]. This network has five nodes and eight arcs, which are numbered from a_1 to a_8. The capacity and lead time of each arc are shown in Table 1.

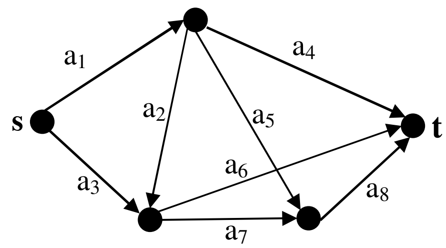

FIGURE 1: Computer network

There are six minimal paths: $mp_1 = \{a_1,a_2\}$, $mp_2 = \{a_1,a_5,a_8\}$, $mp_3 = \{a_1,a_2,a_6\}$, $mp_4 = \{a_1,a_2,a_7,a_8\}$, $mp_5 = \{a_3,a_6\}$, and $mp_6 = \{a_3,a_7,a_8\}$. Given d = 8 and T = 9, Tables 2, 3, and 4 summarize the values of C_{mpj} and T_j for each path mp_j for the different values of C_s, using the algorithm 3.2. Also, below each table we show the value of V_{mp} and the corresponding X vectors, as well as the system reliability $R_{d,Cs,T}$.

Arc	Capacity	Probability	Lead time	Arc	Capacity	Probability	Lead time
a_1	3 2 1 0	0.80 0.10 0.05 0.05	2	a_6	4 3 2 1 0	0.60 0.20 0.10 0.05 0.05	2
a_2	2 1 0	0.80 0.10 0.05 0.05	1	a_7	5 4 3 2 1 0	0.55 0.10 0.10 0.10 0.10 0.05	2
a_3	1 0	0.85 0.10 0.05	3				
a_4	1 0	0.90 0.10	3	a_8	3 2 1 0	0.80 0.10 0.05 0.05	1
a_5	1 0	0.90 0.10	1				

Table 1: Arc data for Fig. 1

mp_j	C_{mpj}	Transmission Time T_j	mp_j	C_{mpj}	Transmission Time T_j
mp_1	1	13	mp_4	3	14
mp_2	1	12	mp_5	2	13
mp_3	3	13	mp_6	2	14

Table 2: Values of C_{mpj} and T_j when $C_s = 1$

According to Table 2, there is no path satisfies constraints of T and C_s. So, $V_{mp} = \Phi$, i.e., the X^1, X^2,…, X^6, does not exist. Therefore, $R_{(8,1,9)} = 0$.

mp_j	C_{mpj}	Transmission Time T_j	mp_j	C_{mpj}	Transmission Time T_j
mp_1	1	9	mp_4	3	10
mp_2	1	8	mp_5	2	9
mp_3	3	9	mp_6	2	10

Table 3: Values of C_{mpj} and T_j when $C_s = 2$

According to Table 3, V_{mp} = {mp_3,mp_5} because C_{mp3} is greater than C_s and C_{mp5} is equal to C_s. Furthermore, both T_3 and T_5 are equal to T. Thus, we have X^3 = (2 2 0 0 0 2 0 0) and X^5 = (0 0 2 0 0 2 0 0). Let $A_3 = \{Y | Y \geq X^3\}$ and $A_5 = \{Y | Y \geq X^5\}$. Then, the system reliability $R_{(8,2,9)} = \Pr\{A_3 \cup A_5\} =$ 0.89145, using the inclusion-exclusion rule, where

$$
\begin{aligned}
\Pr\{A_3\} &= \Pr\{Y \geq (2\,2\,0\,0\,0\,2\,0\,0)\} \\
&= \Pr\{x_1 \geq 2\} \times \Pr\{x_2 \geq 2\} \times \Pr\{x_3 \geq 0\} \times \Pr\{x_4 \geq 0\} \times \Pr\{x_5 \geq 0\} \times \Pr\{x_6 \geq 2\} \times \Pr\{x_7 \geq 0\} \times \Pr\{x_8 \geq 0\} \\
&= 0.90 \times 0.90 \times 1 \times 1 \times 1 \times 0.90 \times 1 \times 1 = 0.729 \\
\Pr\{A_5\} &= \Pr\{Y \geq (0\,0\,2\,0\,0\,2\,0\,0)\} \\
&= \Pr\{x_1 \geq 0\} \times \Pr\{x_2 \geq 0\} \times \Pr\{x_3 \geq 2\} \times \Pr\{x_4 \geq 0\} \times \Pr\{x_5 \geq 0\} \times \Pr\{x_6 \geq 2\} \times \Pr\{x_7 \geq 0\} \times \Pr\{x_8 \geq 0\} \\
&= 1 \times 1 \times 0.95 \times 1 \times 1 \times 0.90 \times 1 \times 1 = 0.855 \\
\Pr\{A_3 \cap A_5\} &= \Pr\{Y \geq (2\,2\,2\,0\,0\,2\,0\,0)\} \\
&= \Pr\{x_1 \geq 2\} \times \Pr\{x_2 \geq 2\} \times \Pr\{x_3 \geq 2\} \times \Pr\{x_4 \geq 0\} \times \Pr\{x_5 \geq 0\} \times \Pr\{x_6 \geq 2\} \times \Pr\{x_7 \geq 0\} \times \Pr\{x_8 \geq 0\} \\
&= 0.90 \times 0.90 \times 0.95 \times 1 \times 1 \times 0.90 \times 1 \times 1 = 0.69255
\end{aligned}
$$

mp_j	C_{mpj}	Transmission Time T_j	mp_j	C_{mpj}	Transmission Time T_j
mp_1	1	8	mp_4	3	9
mp_2	1	7	mp_5	2	8
mp_3	3	8	mp_6	2	9

Table 4: Values of C_{mpj} and T_j when $C_s = 3$

It is clear that V_{mp} = {mp_3,mp_4} because both C_{mp3} and C_{mp4} are equal to C_s. Also, T_3 is less than T and T_4 is equal to T.

Then, we have X^3 = (3 3 0 0 0 3 0 0) and X^4 = (3 3 0 0 0 0 3 3). Let $A_3 = \{Y \mid Y \geq X^3\}$ and $A_4 = \{Y \mid Y \geq X^4\}$. Then, the system reliability is $R_{(8,3,9)} = \Pr\{A_3 \cup A_4\} =$ 0.5888, using the inclusion-exclusion rule, where

$$\Pr\{A_3\} = \Pr\{Y \geq (3\,3\,0\,0\,0\,3\,0\,0)\}$$
$$= \Pr\{x_1 \geq 3\} \times \Pr\{x_2 \geq 3\} \times \Pr\{x_3 \geq 0\} \times \Pr\{x_4 \geq 0\} \times \Pr\{x_5 \geq 0\} \times \Pr\{x_6 \geq 3\} \times \Pr\{x_7 \geq 0\} \times \Pr\{x_8 \geq 0\}$$
$$= 0.80 \times 0.80 \times 1 \times 1 \times 1 \times 0.80 \times 1 \times 1 = 0.512$$
$$\Pr\{A_4\} = \Pr\{Y \geq (3\,3\,0\,0\,0\,0\,3\,3)\}$$
$$= \Pr\{x_1 \geq 3\} \times \Pr\{x_2 \geq 3\} \times \Pr\{x_3 \geq 0\} \times \Pr\{x_4 \geq 0\} \times \Pr\{x_5 \geq 0\} \times \Pr\{x_6 \geq 0\} \times \Pr\{x_7 \geq 3\} \times \Pr\{x_8 \geq 3\}$$
$$= 0.80 \times 0.80 \times 1 \times 1 \times 1 \times 1 \times 0.75 \times 0.80 = 0.384$$
$$\Pr\{A_3 \cap A_4\} = \Pr\{Y \geq (3\,3\,0\,0\,0\,3\,3\,3)\}$$
$$= \Pr\{x_1 \geq 3\} \times \Pr\{x_2 \geq 3\} \times \Pr\{x_3 \geq 0\} \times \Pr\{x_4 \geq 0\} \times \Pr\{x_5 \geq 0\} \times \Pr\{x_6 \geq 3\} \times \Pr\{x_7 \geq 3\} \times \Pr\{x_8 \geq 3\}$$
$$= 0.80 \times 0.80 \times 1 \times 1 \times 1 \times 0.80 \times 0.75 \times 0.80 = 0.3072$$

5. DISCUSSION

Algorithm 3.2 needs O(mn) time to generate all lower boundary points for (d,C_s,T) in the worst case, where n is the number of arcs and m is the number of minimal paths. Algorithm 3.3 needs $O(m^2n)$ time to evaluate the system reliability in the worst case [8], using the inclusion-exclusion rule. Therefore, the total time needed by the algorithm is O(mn) + $O(m^2n)$ to calculate the system reliablity $R_{(d,Cs,T)}$ in the worst case. In comparison with the algorithm presented in [8] to evaluate reliability under the constraint T, the presented algorithm in this paper needs the same time (O(mn) + $O(m^2n)$) to evaluate reliability under C_S and T constraints.

6. CONCLUSIONS and FUTURE WORK

A new definition of the system reliability of a flow network to a given demand d has been presented, which takes into account both the required system capacity (C_s) and the transmission time (T). In addition, an algorithm has been presented for the calculation of $R_{(d,Cs,T)}$. The algorithm is based on the use of minimal paths to generate all lower boundary points for (d,C_s,T), and to then calculate the system reliability $R_{(d,Cs,T)}$ using the inclusion-exclusion rule.

Finally, we have illustrated the use of the proposed algorithm by calculating the reliability of a flow network for a given network taken from the literature.

The algorithm has proved to be efficient and may be used to compute the system reliability of a multicommodity flow network.

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