Verification of the Thermal Buckling Load in Plates Made of Functional Graded Materials

Hamid Mozafari mozafari.h@gmail.com
Faculty of Mechanical Engineering
Technical University of Malaysia-UTM
81310 UTM, Skudai, Johor, Malaysia

Amran Ayob amran@fkm.utm.my
Faculty of Mechanical Engineering
Technical University of Malaysia-UTM
81310 UTM, Skudai, Johor, Malaysia

Amran Alias amran_al @fkm.utm.my
Faculty of Mechanical Engineering
Technical University of Malaysia-UTM
81310 UTM, Skudai, Johor, Malaysia

Abstract

In this study, thermal buckling of thin plate made of Functionally Graded Materials (FGM) with linearly varying thickness is considered. The material properties are also graded in the thickness direction according to a simple power law distribution in which the properties are stated in terms of the volume fractions of the constituents. All edges of the plate are simply supported. The equilibrium and stability equations of a FGM plate under thermal loads can be derived based on higher order plate theories via variation formulation, and are then used to determine the governing deferential equation of the plate and the pre-buckling forces. The buckling analysis of a FGM plate is conducted by assuming a uniform temperature rise, temperature gradient through the thickness, and linear temperature variation in the thickness. Closed–form solutions are obtained the buckling load defined in a weighted residual approach. In a special case the obtained results are compared with the results of FGM plates with uniform thickness. The influences of the plate thickness variation and the edge ratio on the critical loads are investigated. Different gradient exponent *k*, different geometries and loading conditions were studied.

Keywords: Thermal buckling; FGM plates; Thin plate; Higher Order plate theories; Variable thickness plate.

1. INTRODUCTION

Functionally graded materials (FGMs) have received considerable attention in many engineering applications since they were first reported in 1984 in Japan [23]. The main advantage of such materials is the possibility of tailoring desired properties to needs. Obviously, FGM's can be used in a variety of applications which have made them very attractive. Theories of plates and shells have already been applied to high extent, and there are many text books available, such as [1- 3]. Later on, the concept of FGM was proposed in [4] and [5]. The main advantage of FGMs is their high resistance to environments with extremely high temperature and extreme changes in temperature. Ceramic due to low thermal conductance constituents causes resistance to high temperature. One of the main applications of Functionally Graded Materials is their use in power reactors, electronic and magnetic sensors, medical engineering of artificial bones and teeth, chemical industry and in new technologies such as ceramic engines and as resistant covers and protection against corrosion.

Chi and Chung [9, 10] examined the mechanical behavior of FGM plates under transverse load. Najafizadeh and Eslami [14] studied the buckling behavior of circular FGM plates under uniform radial compression. Shariat and Eslami [16] investigated thermal buckling of imperfect FGM plates. Huang and Chang [13] carried out studies on corner stress singularities in an FGM thin plate. Nonlinear analysis, such as nonlinear bending, nonlinear vibration and post-bucking analysis of homogeneous isotropic or FGM plates and shells can be found in the articles by Sundararajan et al. [17], Chen et al. [7], Hsieh and Lee [12], Ghannadpour and Alinia [11]. Further research can be found in the articles by Navazi et al. [15], Woo et al. [18], Chen and Tan [8] and Li et al. [20]. Morimoto et al. [19] and Abrate [6] noticed that there is no stretching–bending coupling in constitutive equations if the reference surface is properly selected. Classical nonlinear laminated plate theory and the concept of physical neutral surface are employed to formulate the basic equations of the FGM thin plate. Da-Guang Zhanga and You-He Zhou studied functionally graded materials as thin plates in 2008 [27], whereas Wu [21] has examined the effect of shear deformation on the thermal buckling of FGM plates. Chen and Liew [22] have examined the buckling of rectangular FGM plates subjected to in-plane edge loads. Based on third order shear deformation theory, Shariat and Eslami [27] studied the buckling of *thick* functionally graded material under mechanical and thermal load and Javaheri and Eslami [28] studied the buckling of functionally graded plate under in-plane compressive loading based on classical plate theory. Previous studies reported that critical buckling temperature differences for the functionally graded plates are generally lower than the corresponding values for homogeneous plates. They used classical plate theory for the buckling analysis of functionally graded plates under in–plane compressive loading.

In the present study, equilibrium and stability equations for functionally graded *thin* plates are derived based on higher order shear deformation plate theory. The resulting equations are employed to obtain closed–form solutions for the critical buckling loads. In order to establish the fundamental system of equations for the buckling analysis, it is assumed that the non-homogeneous mechanical properties of the material are given by a power law formulated in Cartesian coordinates.

2. FGM PLATE AND ITS PROPERTIES

Consider a FG thin plate made from a mixture of ceramics and metals and subjected to a kind of thermal load. The plate coordinate system (*x, y, z*) is chosen such that, *x* and *y* are in-plane coordinates and *z* is in the direction through the thickness and normal to the middle plane. The corresponding displacements in the x-, y- and z-directions are designated by *u*, *v* and *w*, respectively. The origin of the coordinate system is located at the corner of the plate on the middle plane. The plate side lengths in the *x*- and y-directions are designated as *a*, and *b* respectively. The thickness of the plate, *h*, varies the *x* and *y* directions such that (see Fig. 1);

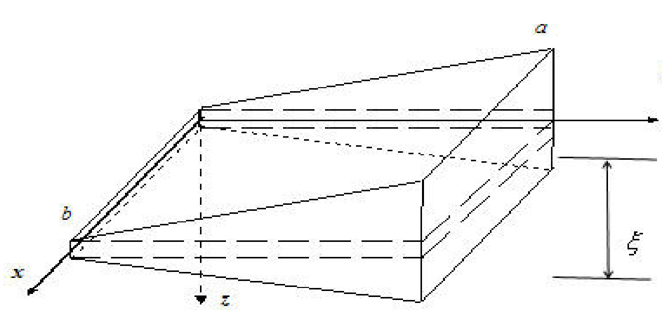

FIGURE 1: Geometry and coordinate system of rectangular plate (a ×b)

h = h(x) = $\xi = c_1 x + c_2$ & h = h(y) = $\xi = c_1 y + c_2$. (1)

in which ξ is a general parameter indicating the thickness change in either of x or y directions, c_2 is the nominal thickness of the plate at the origin and c_1 is a variable parameter called the non-dimensional parameter. When c_1=0, it means that the plate has a constant thickness. When x = 0, one has $\xi = c_2 = h$ and for the case of $x = a$, $\xi(a) = c_1 a + c_2$

To get the change in through thickness properties, it is assumed that the plate composition is varied from the outer (top) to the inner (bottom) surface; i.e. the outer surface of the plate is ceramic rich whereas the inner surface is metal-rich. The material properties of the FGM plate, are such that the coefficient of thermal expansion, α, modulus of elasticity E, and coefficient of thermal conduction k are assumed to be functions of the constituent materials, while the Poisson's ratio v is assumed to be constant across the plate thickness such that:

$$E(z) = E_c V_c + E_m (1 - V_{c)},$$
$$\alpha(z) = \alpha_c V_c + \alpha_m (1 - V_c),$$
$$K(z) = K_c V_c + K_m (1 - V_c), \quad (2)$$
$$v(z) = v,$$

where subscripts m and c refer to the metal and ceramic constituents, respectively. The volume fractions of ceramic v_c and metal v_m are related by

$$V_c = (z/h + 1/2)^k, \quad k \geq 0, \quad k = \infty,$$
$$V_m(z) + V_c(z) = 1, \quad (3)$$

where volume fraction exponent k dictates the material metal-ceramic variation profile through the plate thickness. k assumes values greater than or equal to zero. K = 0 represents a fully ceramic plate. From Eqns. (2) and (3) material properties of the FGM plate are determined, which are the same as the equations proposed by many references.

$$E(z) = E_m + E_{cm} (z/h + 1/2)^k,$$
$$\alpha(z) = \alpha_m + \alpha_{cm} (z/h + 1/2)^k, \quad (4)$$
$$K(z) = K_m + K_{cm} (z/h + 1/2)^k,$$
$$v(z) = v,$$

in which;

$$E_{cm} = E_c - E_m, \qquad \alpha_{cm} = \alpha_c - \alpha_m, \qquad K_{cm} = K_c - K_m, \quad (5)$$

3. BASIC AND EQUILIBRIUM EQUATIONS

The higher order plate theories which is considered in the present work is based on the assumption of the displacement field in the following form:

$$u(x, y, z) = u_0(x, y) - z w_{0,x,}$$

$$v(x,y,z) = v_0(x,y) - zw_{0,y,} \quad (6)$$

$$w(x,y,z) = w_0(x,y)$$

in which *u*, *v*, *w* are the total displacement and (u_0, v_0, w_0) are the mid-plane displacements in the *x*, *y* and *z* directions, respectively. For the thin plate i.e. (*h*/*b*)≤(1/20), where *h* and *n* are the thickness and smaller edge side of the plate, respectively.
Hook's law for a plate with thermal effects is defined as:

$$\bar{\sigma}_{xx} = \frac{E(z)}{1-v^2}\left[\bar{\varepsilon}_{xx} + v\bar{\varepsilon}_{yy} - (1+v)\alpha T\right],$$
$$\bar{\sigma}_{yy} = \frac{E(z)}{1-v^2}\left[\bar{\varepsilon}_{yy} + v\bar{\varepsilon}_{xx} - (1+v)\alpha T\right], \quad (7)$$
$$\bar{\sigma}_{xy} = \frac{E(z)}{2(1+v)}\bar{\gamma}_{xy}$$

The plate is assumed to be comparatively thin and according to the Love-Kirchhoff assumption, planes which are normal to the median surface are assumed to remain plane and normal during deformation, thus out-of-plane shear deformations $(\gamma_{xz}, \gamma_{yz})$ are disregarded. Strain components at distance z from the middle plane are then given by:

$$\bar{\varepsilon}_{xx} = \varepsilon_{xx} + zk_{xx},$$
$$\bar{\varepsilon}_{yy} = \varepsilon_{yy} + zk_{yy}, \quad (8)$$
$$\bar{\gamma}_{xy} = \gamma_{xy} + 2zk_{xy.}$$

Here, $\varepsilon_{xx}, \varepsilon_{yy}, \gamma_{xy}$ denote the corresponding quantities at points on the mid-plane surface only, and k_{xx}, k_{yy}, k_{xy} are the curvatures which can be expressed in term of the displacement components. The relations between the mid-plane strains and the displacement components according to the Sander's assumption are;

$$\varepsilon_{xx} = u_{,x} + \frac{1}{2}w_{,x}^2,$$
$$\varepsilon_{yy} = v_{,y} + \frac{1}{2}w_{,y}^2, \quad (9)$$
$$\gamma_{xy} = u_{,y} + v_{,x} + w_{,x}w_{,y}.$$

and

$$k_{xx} = -w_{,xx}, \qquad k_{yy} = -w_{,yy}, \qquad k_{xy} = -w_{,xy} \quad (10)$$

Substituting Eqns. (9) and (10) into Eqns. (8), the following expressions for the strain components are obtained:

$$\bar{\varepsilon}_{xx} = u_{,x} + \frac{1}{2}w_{,x}^2 - zw_{,xx}, \quad (11)$$
$$\bar{\varepsilon}_{yy} = v_{,y} + \frac{1}{2}w_{,y}^2 - zw_{,yy},$$
$$\bar{\gamma}_{xy} = u_{,y} + v_{,x} + w_{,x}w_{,y} - 2zw_{,xy}.$$

A loaded plate is in equilibrium if its total potential energy *V* remains stationary (δV =0), and *V* is stationary if the integrand in expression for *V* satisfies the Euler equations.

The total potential energy *V* of a plate subjected to thermal loads is defined as:

$$V = U_m + U_b + U_c + U_T, \tag{12}$$

where U_m is the membrane strain energy, U_b is the bending strain energy, U_c is the coupled strain energy, and U_T is the thermal strain energy. The strain energy for thin plate based on classical plate theory is defined as;

$$U = \frac{1}{2}\iiint\left[\bar{\sigma}_{xx}\left(\bar{\varepsilon}_{xx} - \alpha T\right) + \bar{\sigma}_{yy}(\bar{\varepsilon}_{yy} - \alpha T) + \bar{\tau}_{xy}\bar{\gamma}_{xy}\right]dxdydz. \tag{13}$$

Substituting Eqns. (7) and (8) into Eqn. (13), and integrating with respect to *z* from- ξ */2 to* ξ */2,* the total potential energy results in;

$$V = \iint F dxdy. \tag{14}$$

where, the function *F* is;

$$\begin{aligned} F = {} & \frac{A}{2(1-v^2)}\left[\varepsilon_{xx}^2 + \varepsilon_{yy}^2 + 2v\varepsilon_{xx}\varepsilon_{yy} + \frac{1-v}{2}\gamma_{xy}^2\right] + \\ & \frac{C}{2(1-v^2)}\left[k_{xx}^2 + k_{yy}^2 + 2vk_{xx}k_{yy} + 2(1-v)k_{xy}^2\right] + \\ & \frac{B}{2(1-v^2)}\left[\varepsilon_{xx}k_{xx} + \varepsilon_{yy}k_{yy} + v(\varepsilon_{xx}k_{yy} + \varepsilon_{yy}k_{xx}) + (1-v)\gamma_{xy}k_{xy}\right] - \\ & \frac{1}{1-v}\left[\Theta(\varepsilon_{xx} + \varepsilon_{yy}) + \Phi(k_{xx} + k_{yy}) - \psi\right] \end{aligned} \tag{15}$$

where

$$\begin{aligned} A &= \int_{-\xi/2}^{\xi/2} E(z)dz = E_m\xi + E_{cm}\frac{\xi}{k+1}, \\ B &= \int_{-\xi/2}^{\xi/2} E(z)zdz = E_{cm}\frac{k\xi^2}{(2k+2)(k+2)}, \\ C &= \int_{-\xi/2}^{\xi/2} E(z)z^2dz = E_m\frac{\xi^3}{12} + E_{cm}\xi^3\left[\frac{1}{k+3} - \frac{1}{k+2} + \frac{1}{4k+4}\right], \\ (\Theta, \Phi) &= \int_{\xi/2}^{\xi/2}(1, z)E(z)\alpha(z)T(x, y, z)dz, \\ \psi &= \int_{\xi/2}^{\xi/2} E(z)\alpha^2(z)T^2(x, y, z)dz. \end{aligned} \tag{16}$$

The total potential energy is a function of the displacement components and their derivatives. Hence, minimization of total potential energy in terms of the function *F* yields the following Euler Equations:

$$\begin{aligned} & \frac{\partial F}{\partial u} - \frac{\partial}{\partial x}.\frac{\partial F}{\partial u_{,x}} - \frac{\partial}{\partial y}.\frac{\partial F}{\partial u_{,y}} = 0 \\ & \frac{\partial F}{\partial v} - \frac{\partial}{\partial x}.\frac{\partial F}{\partial v_{,x}} - \frac{\partial}{\partial y}.\frac{\partial F}{\partial v_{,y}} = 0 \\ & \frac{\partial F}{\partial w} - \frac{\partial}{\partial x}.\frac{\partial F}{\partial w_{,x}} - \frac{\partial}{\partial y}.\frac{\partial F}{\partial w_{,y}} + \frac{\partial^2}{\partial x^2}.\frac{\partial F}{\partial w_{,xx}} + \frac{\partial^2}{\partial x\partial y}.\frac{\partial F}{\partial w_{,xy}} + \frac{\partial^2}{\partial y^2}.\frac{\partial F}{\partial w_{,yy}} = 0 \end{aligned} \tag{17}$$

Substituting Eqns. (9) and (10) into Eqn. (15) and using Eqns. (17), the equilibrium equations for general rectangular plate made of functionally graded material are given by;

$$N_{x,x} + N_{xy,y} = 0$$

$$N_{xy,x} + N_{y,y} = 0 \tag{18}$$

$$M_{x,xx} + 2M_{xy,xy} + M_{y,yy} + N_x w_{,xx} + 2N_{xy} w_{,xy} + N_y w_{,yy} + P_n = 0$$

where stress resultant N_i, M_i are given by:

$$(N_{i,} M_i) = \int_{\xi/2}^{\xi/2} (1, z).\overline{\sigma}_i .dz, \qquad i = x, y, xy \tag{19}$$

By substituting Eq. (7) into Eq. (19), one can arrive to the following constitutive relation is obtained;

$$(N_{x,} M_x) = \frac{1}{1-v^2}\left[(A, B)(\varepsilon_{xx} + v\varepsilon_{yy}) + (B, C)(k_{xx} + vk_{yy}) - (1+v)(\Theta, \Phi)\right],$$
$$(N_{y,} M_y) = \frac{1}{1-v^2}\left[(A, B)(\varepsilon_{yy} + v\varepsilon_{xx}) + (B, C)(k_{yy} + vk_{xx}) - (1+v)(\Theta, \Phi)\right], \tag{20}$$
$$(N_{xy,} M_{xy}) = \frac{1}{2(1+v)}\left[(A, B)\gamma_{xy} + 2(B, C)k_{xy}\right]$$

4. PLATE STABILITY EQUATIONS

Stability equations of thin plates are derived using the energy method. If V is the total potential energy of the plate, then expanding V about the equilibrium state using Taylor series yields;

$$\Delta V = \delta V + \frac{1}{2!}\delta^2 V + \frac{1}{3!}\delta^3 V + ... \tag{21}$$

The first variation δV is associated with the state of equilibrium. The stability of the plate in the neighborhood of equilibrium condition may be determined by the sign of second variation. The condition $\delta^2 V = 0$ is used to derive the stability equations for buckling problems [16]. Assume that $\hat{u}_i$ denotes the displacement component of the equilibrium state and $\delta\hat{u}_i$ the virtual displacement corresponding to a neighboring state. Denoting $\overline{\delta}$ the variation with respect to $\hat{u}_i$, the following rule, known as the Trefftz rule, is adopted for the determination of the critical load. The external load acting on the plate is considered to be the critical buckling load if the following variation equation is satisfied $\overline{\delta}(\delta^2 V) = 0$. The state of primary equilibrium of a rectangular plate under general loading is designated by u_0, v_0, w_0. In deriving the stability equations, virtual displacements are defined as:

$$u \to u_0 + u_1,$$
$$v \to v_0 + v_1, \tag{22}$$
$$w \to w_0 + w_1,$$

where u_1, v_1, w_1 are the virtual displacement increments. Substituting Eqns. (22) into Eqn. (15) and collecting the second–order terms, the second variation of the potential energy are obtained as;

$$\frac{1}{2}\delta^2 V = \iint \Bigg\{ \frac{A}{2(1-v^2)}\left[u_{1,x}^2 + v_{1,y}^2 + 2vu_{1,x}v_{1,y} + \frac{1-v}{2}(u_{1,y}+v_{1,x})^2 \right] - \frac{B}{1-v^2}\left[u_{1,x}w_{1,xx} + v_{1,y}w_{1,yy} + v(u_{1,x}w_{1,yy} + v_{1,y}w_{1,xx}) + (1-v)(u_{1,y}+v_{1,x})w_{1,xy} \right] + \frac{C}{2(1-v^2)}\left[w_{1,xx}^2 + w_{1,yy}^2 + 2vw_{1,xx}w_{1,yy} + 2(1-v)w_{1,xy}^2 \right] + \frac{1}{2}\left[N_x^0 w_{1,x}^2 + 2N_{xy}^0 w_{1,x}w_{1,y} + N_y^0 w_{1,y}^2 \right] \Bigg\} dxdy \tag{23}$$

Applying the Euler equations (17) to the functional of Eq. (23), the stability equations are obtained as;

$$\begin{aligned} &N_{x1,x} + N_{xy1,y} = 0 \\ &N_{xy1,x} + N_{y1,y} = 0 \\ &M_{x1,xx} + 2M_{xy1,xy} + M_{y1,yy} + (N_x^0 w_{1,xx} + 2N_{xy}^0 w_{1,xy} + N_y^0 w_{1,yy}) = 0 \end{aligned} \tag{24}$$

where

$$\begin{aligned} &(N_{x1}, M_{x1}) = \frac{1}{1-v^2}\left[(A,B)(u_{1,x} + w_{1,y}) - (B,C)(w_{1,xx} + vw_{1,yy}) \right], \\ &(N_{y1}, M_{y1}) = \frac{1}{1-v^2}\left[(A,B)(v_{1,x} + vu_{1,x}) - (B,C)(w_{1,yy} + vw_{1,xx}) \right], \\ &(N_{xy1}, M_{xy1}) = \frac{1}{2(1+v)}\left[(A,B)(u_{1,y} + v_{1,x}) - 2(B,C)w_{1,xy} \right], \\ &N_x^0 = \frac{1}{1-v^2}\left[A(u_{0,x} + w_{0,y}) - B(w_{0,xx} + vw_{0,yy}) \right] - \frac{\Theta}{1-v}, \\ &N_y^0 = \frac{1}{1-v^2}\left[A(v_{0,y} + vu_{0,x}) - B(w_{0,yy} + vw_{0,xx}) \right] - \frac{\Theta}{1-v}, \\ &N_{xy}^0 = \frac{A}{2(1+v)}(u_{0,y} + v_{0,x}) - \frac{B}{1+v}w_{0,xy}. \end{aligned} \tag{25}$$

4.1 Governing Differential Equation for FGM

By substituting Eq. (25) into Eq. (24), the stability equations in terms of displacement components become;

$$\begin{aligned} &A_{,x}(u_{1,x} + vv_{1,y}) + A(u_{1,xx} + vv_{1,xy}) - B_{,x}(w_{1,xx} + vw_{1,yy}) - B(w_{1,xxx} + vw_{1,xyy}) \\ &\quad + \frac{A(1-v)}{2}(u_{1,yy} + v_{1,xy}) - B(1-v)w_{1,xy} = 0, \\ &\frac{1-v}{2}A_{,x}(u_{1,y} + v_{1,x}) + A\frac{1-v}{2}(u_{1,xy} + v_{1,xx}) - (1-v)B_{,x}w_{1,xy} - (1-v)Bw_{1,xxy} \\ &\quad + A(v_{1,yy} + vu_{1,xy}) - B(w_{1,yyy} + vw_{1,xxy}) = 0, \end{aligned} \tag{26}$$

$$B_{,xx}(u_{1,x} + vv_{1,y}) + 2B_{,x}(u_{1,xx} + vv_{1,xy}) + B(u_{1,xxx} + vv_{1,xxy}) + B(v_{1,yyy} + vu_{1,xyy})$$

$$+(1-v)B_{,x}(u_{1,yy}+v_{1,xy})+(1-v)B(u_{1,xyy}+v_{1,xxy})-C_{,xx}(w_{1,xx}+vw_{1,yy})$$

$$-2C_{,x}(w_{1,xxx}+vw_{1,xyy})-2C_{,x}(1-v)w_{1,xyy}-2(1-v)Cw_{1,xxyy}-C(w_{1,xxxx+}$$

$$vw_{1,xxyy})-C(w_{1,yyyy}+vw_{1,xxyy})+(1-v^2)\left[N_x^0 w_{1,xx}+N_y^0 w_{1,yy}+2N_{xy}^0 w_{1,xy}\right]=0$$

In the next step variables *u*, *v* are eliminated in above relation, then the equations of stability Eqn. (26) can be merged into one equation in terms of deflection component *w* and pre-buckling forces only for linear thickness variation, as:

$$(\frac{B^2}{A}-C)\Delta\Delta w_1+(3\frac{B}{A}B_{,x}-2C_{,x})\frac{\partial}{\partial x}\Delta w_1+(\frac{B}{A}B_{,xx}+\frac{B_{,x}^2}{A}-C_{,xx})(w_{1,xx}+$$
$$vw_{1,yy})+(1-v^2)(N_x^0 w_{1,xx}+N_y^0 w_{1,yy}+2N_{xy}^0 w_{1,xy})=0. \quad (27)$$

where

$$\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}. \quad (28)$$

4.2 Solution Method

The method of solving Eq. (27) is based on the series expansion developed by Galerkin [16]. It was originally proposed by Bubnov and sometimes is referred to as the Bubnov-Galerkin method. If the FGM plate is simply supported at all four edges, then the boundary condition are:

$w_1=0,\quad w_{1,xx}=0\quad$ at $\quad x = 0, a$

$w_1=0,\quad w_{1,yy}=0\quad$ at $\quad x = 0, b \quad (29)$

The proposed deflection function w_1 for this case is assumed to be in the following series form;

$$w_1=B_{mn}\sin(m\pi x/a)\sin(n\pi y/b), \qquad (m, n) = 1, 2, 3,\ldots \quad (30)$$

where B_{mn} are constant coefficients, and *m*, *n* are the half wave numbers in the *x*, *y* directions, respectively.

In this study, in order to determine the critical load, the Galerkin method is used. According to this method,

$$\iint_\Omega \phi(w)R(x,y)dxdy=0, \quad (31)$$

in which R(*x*, *y*) is the residue function and $\phi(w)$ is the weight function.

5. THERMAL BUCKLING ANALYSIS

Consider a plate made of functionally graded material with simply supported edge conditions and subjected to an induced in-plane loading in two directions, as shown in Fig. 1. To obtain the critical thermal loading, the pre-buckling forces should be found. Solving the membrane form of equilibrium equations, results in the following force resultants.

$$N_{x_c}=\frac{E_1}{1-\upsilon_0^2}[\Delta T(E_m\alpha_m+\frac{1}{n+1}(E_m\alpha_{cm}+E_{cm}\alpha_m)+\frac{1}{2n+1}E_{cm}\alpha_{cm})](\frac{1+\upsilon_0}{E_1}c_1x-$$
$$\frac{1+\upsilon_0}{2E_1}c_1a)-\frac{1}{1-\upsilon_0}[\Delta T(E_m\alpha_m+\frac{1}{n+1}(E_m\alpha_{cm}+E_{cm}\alpha_m)+\frac{1}{2n+1}E_{cm}\alpha_{cm})](c_1x+c_2)$$

$$N_{y_c} = \frac{E_1}{1-\upsilon_0^2}\upsilon_0[\Delta T(E_m\alpha_m + \frac{1}{n+1}(E_m\alpha_{cm} + E_{cm}\alpha_m) + \frac{1}{2n+1}E_{cm}\alpha_{cm})](\frac{1+\upsilon_0}{E_1}c_1 x -$$
$$\frac{1+\upsilon_0}{2E_1}c_1 a) - \frac{1}{1-\upsilon_0}[\Delta T(E_m\alpha_m + \frac{1}{n+1}(E_m\alpha_{cm} + E_{cm}\alpha_m) + \frac{1}{2n+1}E_{cm}\alpha_{cm})](c_1 x + c_2)$$
$$N_{xy_0} = 0 \tag{32}$$

where R is a non-dimensional constant. The resulting equation then may be solved for a series of selected values of R. The simply supported boundary conditions are defined as

$$w_0(x,0) = w_0(x,b) = w_0(0,y) = w_0(a,y) = 0$$
$$p_y(x,0) = p_y(x,b) = p_x(0,y) = p_x(a,y) = 0$$
$$M_y(x,0) = M_y(x,b) = M_x(0,y) = M_x(a,y) = 0 \tag{33}$$
$$u_0^1(x,0) = u_0^1(x,b) = v_0^1(0,y) = v_0^1(a,y) = 0$$
$$v_1^1(x,0) = u_1^1(x,b) = v_1^1(0,y) = v_1^1(a,y) = 0$$

The following approximate solutions are found to satisfy both the differential equations and the boundary conditions

$$u_0^1 = u_{0mn}\cos\frac{m\pi}{a}x\sin\frac{n\pi y}{b}$$
$$u_1^1 = u_{1mn}\cos\frac{m\pi}{a}x\sin\frac{n\pi y}{b} \qquad m, n = 1, 2, 3\ldots.$$
$$v_0^1 = v_{0mn}\sin\frac{m\pi}{a}x\cos\frac{n\pi y}{b}$$
$$v_1^1 = v_{1mn}\sin\frac{m\pi}{a}x\cos\frac{n\pi}{b}y \tag{34}$$
$$w_0^1 = w_{0mn}\sin\frac{m\pi}{a}x\sin\frac{n\pi y}{b}y$$

where m and n are number of half waves in x and y directions, respectively, and $(u_{0mn}, u_{1mn}, \upsilon_{0mn}, \upsilon_{1mn}, \omega_{0mn})$ are constant coefficients. Substituting Eqns. (34) into the stability equations (24) and using the kinematic and constitutive relations yields a system of five homogeneous equations for $u_{0mn}, u_{1mn}, \upsilon_{0mn}, \upsilon_{1mn},$ and $\omega_{0mn,}$ i. e.

$$[k_{ij}]\begin{pmatrix} u_{0mn} \\ v_{0mn} \\ w_{0mn} \\ u_{1mn} \\ v_{1mn} \end{pmatrix} = 0 \tag{35}$$

in which K_{ij} is a symmetric matrix with the components

$$k_{11} = E_1[(\frac{m\pi}{a})^2 + \frac{1-\upsilon_0}{2}(\frac{n\pi}{b})^2]$$

$$k_{12} = E_1\frac{(1+\upsilon_0)}{2}(\frac{m\pi}{a})(\frac{n\pi}{b})$$

$$k_{14} = (E_2 - \frac{4E_4}{3h^2})[(\frac{m\pi}{a})^2 + \frac{1-\upsilon_0}{2}(\frac{n\pi}{b})^2]$$

$$k_{15} = (\frac{E_2}{2} - \frac{2E_4}{3h^2})(1+\upsilon_0)(\frac{m\pi}{a})(\frac{n\pi}{b})$$

$$k_{21} = k_{12}$$

$$k_{22} = E_1[\frac{1-\upsilon_0}{2}(\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2]$$

$$k_{23} = -\frac{4E_4}{3h^2}[((\frac{m\pi}{a})^2(\frac{n\pi}{b}) + (\frac{n\pi}{b})^3]$$

$$k_{24} = (\frac{E_2}{2} - \frac{2E_4}{3h^2})(1+\upsilon_0)(\frac{m\pi}{a})(\frac{n\pi}{b})$$

$$k_{25} = (2 - \frac{4E_4}{3h^2})[\frac{1-\upsilon_0}{2}(\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2]$$

$$k_{31} = k_{13}$$

$$k_{32} = k_{23}$$

$$\bar{k}_{33} = \frac{16E_7}{9h^4}[(\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2]^2 - (\frac{4E_3}{h^2} - \frac{E_1}{2} - \frac{8E_5}{h^2})$$

$$(1-\upsilon_0)[(\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2] + (1-\upsilon_0^2)]N_{x_0}(\frac{m\pi}{a})^2 + N_{y_0}(\frac{n\pi}{b})^2]$$

$$k_{34} = -(\frac{4E_3}{h^2} - \frac{E_1}{2} - \frac{8E_5}{h^4})(1-\upsilon_0)(\frac{m\pi}{a}) +$$

$$(\frac{16E_7}{9h^4} - \frac{4E_5}{3h^2})[(\frac{m\pi}{a})^3 + (\frac{m\pi}{a})(\frac{n\pi}{b})^2] \quad (36)$$

$$k_{35} = -(\frac{4E_3}{h^2} - \frac{E_1}{2} - \frac{8E_5}{h^4})(1-\upsilon_0)(\frac{n\pi}{b}) + (\frac{16E_7}{9h^4} - \frac{4E_5}{3h^2})$$

$$[(\frac{m\pi}{a})^2(\frac{m\pi}{b}) + (\frac{n\pi}{b})^3]$$

$$k_{41} = -k_{14}$$

$$k_{42} = -k_{24}$$

$$k_{43} = -k_{34}$$

$$k_{44} = (\frac{8E_5}{3h^2} - \frac{16E_7}{9h^4} - E_3)[(\frac{m\pi}{a})^2 + \frac{1-\upsilon_0}{2}(\frac{n\pi}{b})^2] - (\frac{E_1}{2} - \frac{4E_3}{h^2} + \frac{8E_5}{h^4})(1-\upsilon_0)$$

$$k_{45} = (\frac{4E_5}{3h^2} - \frac{E_3}{2} - \frac{8E_7}{9h^2}E_3)(\frac{m\pi}{a})(\frac{n\pi}{b})$$

$$k_{52} = -k_{25}$$

$$k_{53} = -k_{35}$$

$$k_{51} = -k_{15}$$

$$k_{54} = k_{45}$$

$$k_{55} = (\frac{8E_5}{3h^2} - \frac{16E_7}{9h^4} - E_3)[\frac{1-\upsilon_0}{2}(\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2] (\frac{E_1}{2} - \frac{4E_3}{h} + \frac{8E_5}{h^4})(1-\upsilon_0)$$

Substituting pre-bucking forces from Eqs. The relation of K_{33} and setting $|K_{ij}| = 0$ to obtain the nonzero solution, the value of the F_x is found as

$$\Delta T = \frac{2b^2 k_d}{\pi^2(1+\upsilon_0)(c_1 a + 2c_2)\eta^*[(\frac{mb}{a})^2 + n^2]k_c} \tag{37}$$

where

$$\eta^* = [E_m\alpha_m + (E_m\alpha_{cm} + \alpha_m E_{cm})/(n+1) + E_{cm}\alpha_{cm}/(2n+1)] \tag{38}$$

$$k_d = \det|k_{ij}|$$

$$k_c = k_{15}k_{24}k_{42}k_{51} + k_{12}k_{25}k_{44}k_{51} + k_{14}k_{22}k_{45}k_{51} + k_{14}k_{25}k_{41}k_{52} + k_{15}k_{21}k_{44}k_{52}$$
$$k_{15}k_{21}k_{44}k_{52} + k_{11}k_{24}k_{45}k_{52} + k_{15}k_{22}k_{41}k_{54} + k_{11}k_{25}k_{42}k_{54} + k_{12}k_{21}k_{45}k_{54}$$
$$k_{12}k_{24}k_{41}k_{55} + k_{14}k_{21}k_{42}k_{55} + k_{11}k_{22}k_{44}k_{55} - k_{14}k_{25}k_{42}k_{51} - k_{15}k_{22}k_{44}k_{51}$$
$$k_{12}k_{24}k_{45}k_{51} - k_{15}k_{24}k_{41}k_{52} - k_{11}k_{25}k_{44}k_{52} - k_{14}k_{21}k_{45}k_{52} - k_{12}k_{25}k_{41}k_{54}$$
$$-k_{15}k_{21}k_{42}k_{54} - k_{11}k_{22}k_{45}k_{54} - k_{14}k_{22}k_{41}k_{55} - k_{11}k_{24}k_{42}k_{55} - k_{12}k_{21}k_{44}k_{55}$$

In this section, the closed form solutions of Eq. (27) for three types of thermal loading conditions are presented. The plate is assumed to be simply supported in all edges and rigidity fixed against any extension.

Case A. *Uniform Temperature Rise*

The initial uniform temperature of the plate T_i is uniformly raised to a final value T_f, such that the plate buckles. To find the critical buckling temperature difference i.e., $\Delta T_A = T_f - T_i$, the pre- buckling thermal forces, should be found. Solving the membrane form of equilibrium equations i.e., Eq. (18), gives the pre- buckling force resultants as;

$$N_x^0 = -\frac{\Delta T_A G_1}{2(1-v)}(c_1 a + 2c_2),\ N_y^0 = -\frac{\Delta T_A G_1}{2(1-v)}(c_1 a + 2c_2) - \Delta T_A G_1 c_1 x, \qquad N_{xy}^0 = 0. \tag{39}$$

where

$$G_1 = \left[E_m \alpha_m + (E_m \alpha_{cm} + E_{cm}\alpha_m)/(k+1) + E_{cm}\alpha_{cm}/(2k+1)\right], \tag{40}$$

By substituting this type of loading condition into Eqns. (16), one can get;

$$\Theta = \Delta T_A . \xi . G_1 \tag{41}$$

Substituting Eqn. (39) into Eqn. (27), the buckling equation for this type of loading is obtained as;

$$(\frac{B^2}{A} - C)\Delta\Delta w_1 + (3\frac{B}{A}B_{,x} - 2C_{,x})\frac{\partial}{\partial x}\Delta w_1 + (\frac{B}{A}B_{,xx} + \frac{B_{,x}^2}{A} - C_{,xx})(w_{1,xx} + v w_{1,yy})$$
$$-(1-v^2)\Delta T_A G_1 \left[(\frac{c_1 a}{2(1-v)} + \frac{c_2}{1-v}) w_{1,xx} + (c_1 x + \frac{v c_1 a}{2(1-v)} + \frac{c_2}{1-v}) w_{1,yy}) \right] = 0. \tag{42}$$

For the assumed displacement field given by Eqn. (30) the result of Eqns. (31), (42) becomes;

$$\frac{\pi^2 B_{mn}}{a^4 b^4}\int_0^b \int_0^a \{\pi^2 (m^2 b^2 + n^2 a^2)^2 (\tilde{B}^2/\tilde{A} - \tilde{C})(c_1 x + c_2)^3 \sin(m\pi x/a)\sin(n\pi y/b) +$$
$$6(\tilde{B}^2/\tilde{A} - \tilde{C}) m a b^2 (m^2 b^2 + n^2 a^2) c_1 (c_1 x + c_2)^2 \cos(m\pi x/a)\sin(n\pi y/b) +$$
$$6(\tilde{B}^2/\tilde{A} - \tilde{C})(m^2 b^2 + v n^2 a^2) a^2 b^2 (c_1^3 x + c_1^2 c_2)\sin(m\pi x/a)\sin(n\pi y/b) + \tag{43}$$
$$(1-v^2)\Delta T_A G_1 \left[(c_1 a/2(1-v^2) + c_2/(1-v))(m/a)^2 + (c_1 x + v c_1 a/2(1-v) + c_2/(1-v)) \right.$$
$$\left. (n/b)^2 \right] a^4 b^4 \sin(m\pi x/a)\sin(n\pi y/b) \ \} \sin(m\pi x/a)\sin(n\pi y/b) dx dy = 0$$

After carrying out the integration, one would get;

$$\Delta T_A = H . \left[(mb/a)^2 + n^2 \right] \tag{44}$$

in which,

$$H = \frac{\pi^2}{b^2 (1+v)(c_1 a/2 + c_2) G_1}\{(\tilde{C} - \tilde{B}^2/\tilde{A})(c_1^3 a^3/4 + c_1^2 c_2 a^2 + 3 c_1 c_2^2 a/2 + c_2^3)$$
$$+ 6(\tilde{C} - \tilde{B}^2/\tilde{A})\frac{a^2 b^2 (m^2 b^2 + v n^2 a^2)}{\pi^2 (m^2 b^2 + n^2 a^2)^2}(c_1^3 a/2 + c_1^2 c_2) \ \}. \tag{45}$$

where,

$$\tilde{A} = E_m + E_{cm} /(k+1), \qquad \tilde{B} = E_{cm} k /(2k+2)(2k+1), \tag{46}$$
$$\tilde{C} = E_m /12 + E_{cm}\left[1/(k+3) - 1/(k+2) + 1/(4k+4)\right].$$

The critical buckling load ΔT_A^{cr} can be obtained for different values of *m*, *n* such that it minimizes Eq.(44). Apparently, when minimization methods are used, the critical buckling load, ΔT_A^{cr}, is obtained for *m* = *n* = 1, thus;

$$\Delta T_A^{cr} = \frac{\pi^2\left[(b/a)^2+1\right]}{b^2(1+v)(c_1 a/2 + c_2)G_1}\Big\{(\tilde{C} - \tilde{B}^2/\tilde{A})(c_1^3 a^3/4 + c_1^2 c_2 a^2 + 3c_1 c_2^2 a/2 + c_2^3)$$
$$+ 6(\tilde{C} - \tilde{B}^2/\tilde{A})\frac{a^2 b^2 (b^2 + va^2)}{\pi^2 (b^2 + a^2)^2}(c_1^3 a/2 + c_1^2 c_2)\Big\}. \tag{47}$$

When $c_1 = 0$, Eq. (40) represents the critical thermal buckling load, ΔT_A^{cr} of a FGM plate with constant thickness $c_2 = h$, i.e. ;

$$\Delta T_{Ai}^{cr.} = \frac{\pi^2\left[(\frac{b}{a})^2+1\right]}{b^2(1+v)hG_1}\left(\frac{A.C - B^2}{A}\right). \tag{48}$$

The result given in Eq. (41) is exactly the same as the one obtained by reference [9].

Case B. *Linear Temperature variation across the thickness*

For a functionally graded plate, usually the temperature change is not uniform where the temperature level is much higher at the ceramic side than that in the metal side of the plate. In this case, the temperature variation through the thickness is given by;

$$T(z) = \frac{\Delta T_B}{\xi}(z + \frac{\xi}{2}) + T_m \tag{49}$$

in which

$$T\Big|_{z=\frac{\xi}{2}} = T_c,$$
$$T\Big|_{z=-\frac{\xi}{2}} = T_m,$$
$$\Delta T_B = T_c - T_m \tag{50}$$

T_c and T_m denote the temperature level at the top (ceramic side) and the bottom (metal side) surfaces, respectively. The pre-buckling forces now can be obtained by solving the membrane form of equilibrium equations, i.e. Eqn. (18) and this gives;

$$N_x^0 = -\frac{c_1 a/2 + c_2}{1-v}(\Delta T_B G_2 + T_m G_1),$$
$$N_y^0 = -[c_1 x + \frac{v c_1 a}{2(1-v)} + \frac{c_2}{1-v}](\Delta T_B G_2 + T_m G_1), \qquad N_{xy}^0 = 0. \tag{51}$$

in which

$$G_2 = [E_m \alpha_m / 2 + (E_m \alpha_{cm} + E_{cm} \alpha_m)/(k+2) + E_{cm} \alpha_{cm} /(2k+2)], \quad (52)$$

Substituting Eqn. (51) into Eqn. (27), the buckling equation for this case of loading is obtained;

$$(\frac{B^2}{A} - C)\Delta\Delta w_1 + (3\frac{B}{A}B_{,x} - 2C_{,x})\frac{\partial}{\partial x}\Delta w_1 + (\frac{B}{A}B_{,xx} + \frac{B_{,x}^2}{A} - C_{,xx})(w_{1,xx} + v w_{1,yy)}$$
$$- (1-v^2)[(\frac{c_1 a}{2(1-v)} + \frac{c_2}{1-v})w_{1,xx} + (c_1 x + \frac{v c_1 a}{2(1-v)} + \frac{c_2}{1-v})w_{1,yy})](\Delta T_B G_2 + T_m G_1) = 0 \quad (53)$$

Following similar steps to that given in case A, the buckling load for case B is;

$$\Delta T_B^{cr} = \frac{\pi^2[(b/a)^2 + 1]}{b^2(1+v)(c_1 a/2 + c_2)G_2}\{(\tilde{C} - \tilde{B}^2/\tilde{A})(c_1^3 a^3/4 + c_1^2 c_2 a^2 + 3c_1 c_2^2 a/2 + c_2^3)$$
$$+ 6(\tilde{C} - \tilde{B}^2/\tilde{A})\frac{a^2 b^2 (b^2 + v a^2)}{\pi^2 (b^2 + a^2)^2}(c_1^3 a/2 + c_1^2 c_2)\ \} - \frac{T_m G_1}{G_2} \quad (54)$$

When $c_1 = 0$, Eq. (47) is reduced to the critical buckling load, $\Delta T_{B_i}^{cr}$ of a FGM plate, with constant thickness $c_2 = h$, which is;

$$\Delta T_B^{cr} = \frac{\pi^2[(b/a)^2 + 1]}{b^2(1+v)hG_2}(\frac{A.C - B^2}{A}) - \frac{T_m G_1}{G_2} \quad (55)$$

The result given in Eq. (55) is exactly the same as the one obtained by reference.

Case C. *Buckling of FGM plate under Non-linear temperature variation across the Thickness*
In this section, the governing differential equation for the temperature distribution through the thickness is given by one- dimensional Fourier equation under steady state heat condition as;

$$\frac{d}{dz}\left[k(z)\frac{dT}{dz}\right] = 0, \quad (56)$$

where $k(z)$ is the coefficient of thermal conduction. Similar to what was considered for the variation of the elastic modulus and coefficient of thermal expansion, here the coefficient of the heat conduction is also assumed to change according to a power law in terms of *z* as represented by Eq. (49).

By inserting Eq. (49) into Eq. (56) one would get;

$$\frac{d^2 T}{dm^2} + \frac{k K_{cm} m^{k-1}}{K_m + K_{cm} m^k}\frac{dT}{dm} = 0, \quad (57)$$

in which

$$m = \frac{2z + \xi}{2\xi} \quad (58)$$

and the boundary conditions across the plate thickness are;

$$T = T_{c,} \qquad\qquad m = 1, \tag{59}$$

$$T = T_{m,} \qquad\qquad m = 0,$$

The solution of Eq. (50) can be obtained by means of polynomial series. Taking the first seven terms of the series;

$$T = \hat{C}_0 + \hat{C}_1 m + \hat{C}_2 m^2 + \hat{C}_3 m^3 + \hat{C}_4 m^4 + \hat{C}_5 m^5 + \hat{C}_6 m^6, \tag{60}$$

In which $\hat{C}_i$ are constant coefficients to be evaluated. After substituting Eqn. (60) Into Eqn. (57), imposing the boundary conditions and repeating similar above mathematical manipulations, one can get;

$$T(z) = T_m + \frac{\Delta T_c}{\hat{C}_0} L(z). \tag{61}$$

in which

$$\hat{C}_0 = 1 - \frac{K_{cm}}{(k+1)K_m} + \frac{K_{cm}^2}{(2k+1)K_m^2} - \frac{K_{cm}^3}{(3k+1)K_m^3} + \frac{K_{cm}^4}{(4k+1)K_m^4} - \frac{K_{cm}^5}{(5k+1)K_m^5},$$

$$\begin{aligned} L(z) = {} & (\frac{2z+\xi}{2\xi}) - \frac{K_{cm}}{(k+1)K_m}(\frac{2z+\xi}{2\xi})^{k+1} + \frac{K_{cm}^2}{(2k+1)K_m^2}(\frac{2z+\xi}{2\xi})^{2k+1} \\ & - \frac{K_{cm}^3}{(3k+1)K_m^3}(\frac{2z+\xi}{2\xi})^{3k+1} + \frac{K_{cm}^4}{(4k+1)K_m^4}(\frac{2z+\xi}{2\xi})^{4k+1} - \frac{K_{cm}^5}{(5k+1)K_m^5}(\frac{2z+\xi}{2\xi})^{5k+1} \end{aligned} \tag{62}$$

$$\Delta T_c = T_c - T_m$$

The pre-buckling resultants loads for this case can be obtained by solving the membrane effects of the equilibrium equations i.e., (Eq. (18)) which yields;

$$N_x^0 = -\frac{c_1 a/2 + c_2}{1-v}(\Delta T_c G_3 + T_m G_1)$$

$$N_y^0 = -[c_1 x + \frac{v c_1 a}{2(1-v)} + \frac{c_2}{1-v}](\Delta T_c G_3 + T_m G_1)$$

$$N_{xy}^0 = 0 \tag{63}$$

Substituting $T(z)$ in Eqns. (16) and calculating for Θ ;

$$\Theta = T_m \int_{-\xi/2}^{\xi/2} E(z).\alpha(z)dz + \frac{\Delta T_c}{\hat{C}_0}\int_{-\xi/2}^{\xi/2} L(z).E(z).\alpha(z)dz \tag{64}$$

By substituting Eqn. (63) into Eqn. (27) the buckling for this case of loading is obtained. By performing an analysis similar to that done for the case A, the thermal critical buckling load, ΔT_c^{cr} for case C is determined to be.

$$\begin{aligned} \Delta T_C^{cr} = {} & \frac{\pi^2[)b/a)^2+1]}{b^2(1+v)(c_1 a/2 + c_2)G_3}\Big\{(\tilde{C} - \hat{B}^2/\tilde{A})(c_1^3 a^3/4 + c_1^2 c_2 a^2 + 3c_1 c_2^2 a/2 + c_2^3) \\ & + 6(\tilde{C} - \tilde{B}^2/\tilde{A})\frac{a^2 b^2 (b^2 + v a^2)}{\pi^2 (b^2 + a^2)^2}(c_1^3 a/2 + c_1^2 c_2)\ \Big\} - \frac{T_m G_1}{G_3}, \end{aligned} \tag{65}$$

in which

$$G_3 = 1/\hat{C}_0 \{ E_m\alpha_m[1/2 - K_{cm}/(k+2)(k+1)K_m + K_{cm}^2/(2k+2)(2k+1)K_m^2 - K_{cm}^3/(3k+1)(3k+2)K_m^3 + K_{cm}^4/(4k+2)(4k+1)K_m^4 - K_{cm}^5/(5k+2)(5k+1)K_m^5] + (E_m\alpha_{cm} + E_{cm}\alpha_m)[1/(k+2) - K_{cm}/(2k+2)(k+1)K_m + K_{cm}^2/(3k+2)(2k+1)K_m^2 - K_{cm}^3/(4k+2)(3k+1)K_m^3 + K_{cm}^4/(5k+2)(4k+1)K_m^4 - K_{cm}^5/(6k+2)(5k+1)K_m^5] + E_{cm}\alpha_{cm}[1/(2k+2) - K_{cm}/(3k+2)(k+1)k_m + K_{cm}^2/(2k+1)(4k+2)K_m^2 - K_{cm}^3/(5k+2)(3k+1)K_m^3 + K_{cm}^4/(6k+2)(4k+1)K_m^4 - K_{cm}^5/(7k+2)(5k+1)K_m^5] \} \quad (66)$$

When $c_1 = 0$, Eq. (65) will be reduced to the critical buckling load, $\Delta T_{C_i}^{cr}$ of a FGM plate with constant thickness $c_2 = h$, which is;

$$\Delta T_{C_i}^{cr} = \frac{\pi^2 (A.C - B^2)}{b^2(1+v)A.h.G_3}[(b/a)^2 + 1] - \frac{T_m G_1}{G_3} \quad (67)$$

The result given in Eq. (67) is exactly the same as the one obtained by reference

6. RESULTS AND DISCUSSIONS

In this paper, the pre-buckling and critical thermal buckling loads of a thin FGM plate with variable thickness are obtained. The analysis on thickness variation of the plate is carried out for two different types of linear heat conduction variations both in *x*- and *y*-directions. In order to conduct further calculations, a functionally graded material consisting of aluminum and alumina is considered in which the Young's modulus, conductivity, and the coefficient of thermal expansion, are: for aluminum, $K_m = 204 w/mk,\ \alpha_m = 23\times10^{-6}(1/{}^0C)$ $E_m = 70GPa$ and for alumina, $E_c = 380GPa,\ K_c = 10.4 w/mk,\ \alpha_c = 7.4\times10^{-6}(1/{}^0C)$. The Poisson's ratio $\nu_m = \nu_c = 0.3$ are taken for both.

The graphs of critical temperature change ΔT^{cr} versus the aspect ratio b/a , c_1, and volume fraction exponent *k* for two types of linear change of thickness at *x*, *y* directions and three types of thermal loading are shown in Figs. (2- 4). To begin with, the variation of the critical temperature difference ΔT_A^{cr} of FGM plate under uniform temperature rise vs. different geometric parameter (b/a), for different volume fraction exponents is analyzed. The variations are plotted in Fig. (2). By comparing the values of the critical temperature differences ΔT_A^{cr} calculated with using linear change in the plate thickness in the *x*-direction are lower than in *y*-direction. For the plate of FGM material (*k* > 0), the critical temperature difference of buckling for thickness variation in the *y*-direction is higher than in *x*-direction. Therefore, the plate resistance against buckling for all kinds of thermal loads is higher in the *y*-direction when the plate has a variable thickness.

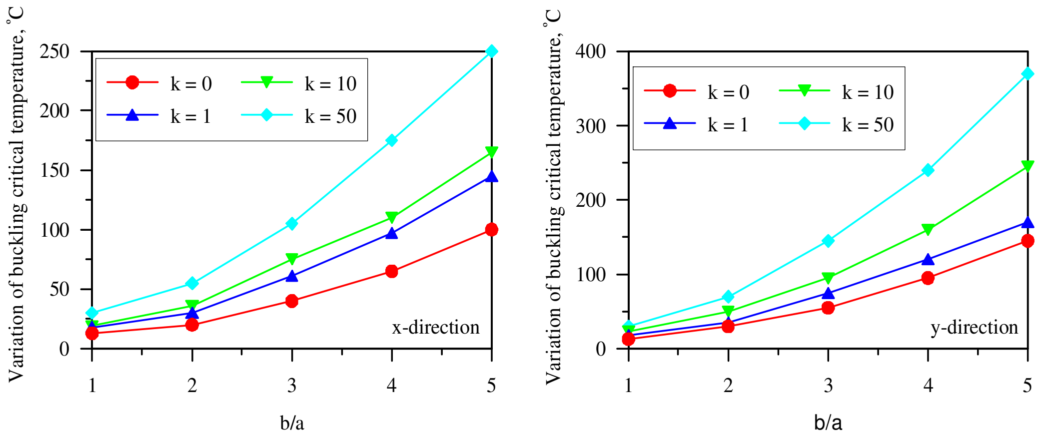

FIGURE 2: Variation of buckling critical temperature (Case A)

Figure 2 is shown the variation of buckling critical temperature gradient against *b*/*a* for different functionally graded material plate with linear thickness change in *x*-direction and *y*-direction under uniform temperature rise.

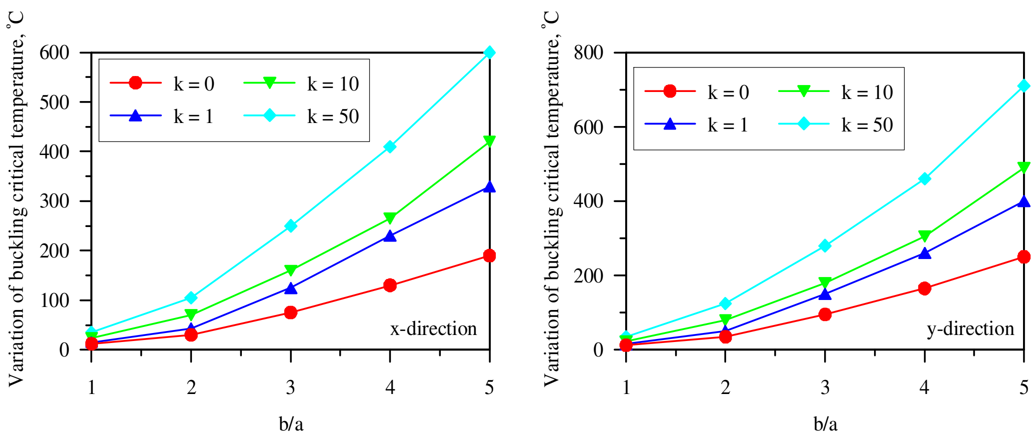

FIGURE 3: Variation of buckling critical temperature (Case B)

Figure 3 is illustrated the variation of buckling critical temperature gradient against material index *k* for different FGM plate with linear thickness change in both *x* and *y* directions under linear temperature across thickness.

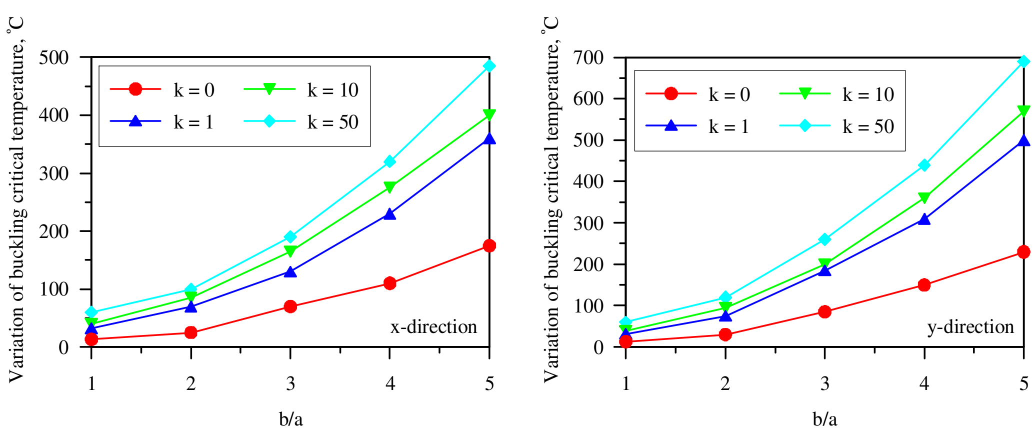

FIGURE 4: Variation of buckling critical temperature (Case C)

Figure 4 is shown the variation of the buckling critical temperature gradient against material index *k* for different FGM plate with linear thickness change in both *x* and *y* directions under non-linear temperature across thickness.

As an overview of all the above cases, one can say that the critical buckling temperature gradient of a homogeneous ceramic plate (k = 0), is higher than the FGM plate. This result is justifiable, because the coefficient of the thermal expansion of ceramic plate is lower than the FGM plate. Referring to Figs. (2- 4) it can be said that the difference between variation of critical buckling temperature gradient of the homogeneous ceramic plate (k = 0) and the FGM plate (k > 0) is not significantly high. Contrary to this, for the other types of loading the difference is much higher; therefore, this type of loading results in a more acceptable low thermal stress distribution in the plate.

In Figs. 2- 4, it is found that the critical temperature difference of FGM plate is higher than that of the fully metallic isotropic plate but lower than that of the fully ceramic isotropic plate. In addition, the critical temperature change decreases as the volume fraction exponent *k* is increased. In all cases, the critical temperature difference increases, when the geometric parameter *b*/*a* is increased.

7. CONCLUSIONS

In the present paper, equilibrium and stability equations for a simply supported rectangular functionally graded plate with its thickness varying along both the *x*- and y-axis as a linear function, under thermal loading are obtained according to the classical plate theory. The critical buckling temperature gradient for three different types of thermal loading is derived using Galerkin method. From the results, primarily one can conclude that the thickness change causes reinforcement of resistance of buckling.

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