Mathematical Derivation of Annuity Interest Rate and its Application

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Abstract

A fundamental task in business for investor or borrower is to know the interest rate of an annuity. In this type of problem, the size of each periodic payment(R), the term(n), and the amount(Sn) or the present value of the annuity(An) are usually given.However, a direct equation representing the Annuity Interest Rate(i) is not available, since an approximate value of the Annuity Interest Rate is obtained by interpolation methodbased on table showing (Sn/R) values. This paper emphasizes the real time computational problem for Annuity interest rate. It has therefore been important to derive an equation for computing the Annuity Interest rate. The evaluation of error analysis has been discussed. The new algorithm saved computational energy by approximately 99.9%than that of the tabulated one.

Keywords: Investment Mathematics, Statistical Toolbox**,** MATLAB Programming.

1. INTRODUCTION

There are many situations in which both businesses and individuals would be faced with either receiving or paying a constant amount for a length of period. When a firm faces a stream of constant payments on a bank loan for a period of time, we call that stream of cash flows an annuity.

An annuity is a series of periodic payments, usually made in equal amounts. The payments are computed by the compound interest method[1] and are made at equal intervals of time. Individual investors may make constant payments on their home or car loans, or invest a fixed amount year after year to save for their retirement. Any financial contract that calls for equally spaced and level cash flows over a finite number of periods is called an annuity. If the cash flow payments continue forever, the contract is called perpetuity. Constant cash flows that occur at the end of each period are called ordinary annuities.

2. THE AMOUNT OF AN ANNUITY

In Business, the amount of an annuity is the final value at the end of the term of the annuity. To derive the formula for the amount of an ordinary annuity,
let:

R is the size of each regular payment.
i is the interest rate per conversion period.
n is the number of payments during the term of an annuity.
S_n is the amount of an ordinary annuity.

Then:

The amount of an ordinary annuity is given by:

Term 0	1	2	---------------	(n-2)	(n-1)	n	Amount of annuity(S_n)
Payment						R	**R +**
					R		**R(1+i) +**
				R			**R(1+i)^2 +**
			---------------				**--------- +**
		R					**R(1+i)$^{n-2}$+**
	R						**R(1+i)$^{n-1}$**
i annuity interest rate per conversion							

$$S_n = R + R(1+i) + R(1+i)^2 + \cdots + R(1+i)^{n-2} + R(1+i)^{n-1} \qquad (1)$$

Multiply both sides by (1+i), we have:

$$S_n(1+i) = R(1+i) + R(1+i)^2 + \cdots + R(1+i)^{n-1} + R(1+i)^n \qquad (2)$$

Subtracting Eq.(2) from Eq.(1), we get:

$$S_n(1+i) - S_n = R(1+i)^n - R$$

$$S_n(1+i-1) = R[(1+i)^n - 1]$$

$$S_n = \mathrm{R}\frac{[(1+i)^n - 1]}{i} \qquad (3)$$

3. THE PRESENT VALUE OF AN ANNUITY

The present value of an annuity is the value at the beginning of the term of the annuity. The present value of an annuity can be derived by the same way to get the following formula:

$$A_n = \mathrm{R}\frac{[1 - (1+i)^{-n}]}{i} \qquad (4)$$

Where:

An is the present value of an ordinary annuity.

4. ANNUITY INTEREST RATE PER CONVERSION(i)

The annuity equation (Eq.3 or Eq.4)can also be used to the find the interest rate or discount rate for an annuity.To determine an accurate valueof the Annuity interest rate instead of using a trial-and-error approach, we need to solve the equation for the unknown value i as follow:

a) When the Amount is Known (S_n)

(1) Two_Term Simplification

To find the annuity interest rate when the amount is known, use the Eq.(3) as follow:

$$\frac{S_n}{R}\mathrm{i} = (1+i)^n - 1 \qquad (5)$$

From eq.(5), the term $(1+i)^n$ can be simplified using binomial theorem, since it can obtain the binomial series which is valid for any real number n if $|x| < 1$ as follow:

$$(1+x)^n = 1 + n\,x + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots] \qquad (6)$$

The term $(1+i)^n$ can be rewritten in the following form:

By replacing x by i, we have:

$$(1+i)^n = 1 + n\,i + \frac{n(n-1)}{2!}i^2 + \frac{n(n-1)(n-2)}{3!}i^3 + \cdots] \qquad (7)$$

From eq.(5) & the two ith term expansion of eq.(7), We have:

$$\frac{S_n}{R}\mathrm{i} = (1+i)^n - 1$$

$$= -1 + (1+i)^n$$

$$= -1 + 1 + n\,i + \frac{n(n-1)}{2!}i^2$$

$$= n\,i + \frac{n(n-1)}{2}i^2$$

Dividing both sides by *i*, we get:

$$\frac{S_n}{R} = n + \frac{n(n-1)}{2}i$$

$$2\left(\frac{S_n}{R}-n\right)=n(n-1)i$$

$$\therefore i=\frac{2\left(\frac{S_n}{R}-n\right)}{n(n-1)} \quad ,\forall \quad \frac{S_n}{R}>n \qquad (8)$$

Therefore:

Equation(8) represents the annuity interest rate equation for computing *i* after the two ith term expansion.

(2) Three_Term Simplification

From eq.(5) & the three ith term expansion of eq.(7), We have:

$$\frac{S_n}{R}i = (1+i)^n-1$$
$$= -1+(1+i)^n$$
$$= -1+1+n\,i+\frac{n(n-1)}{2!}i^2+\frac{n(n-1)(n-2)}{3!}i^3$$
$$= n\,i+\frac{n(n-1)}{2}i^2+\frac{n(n-1)(n-2)}{6}i^3$$

Dividing both sides by *i*, we get:

$$\frac{S_n}{R}=n+\frac{n(n-1)}{2}i+\frac{n(n-1)(n-2)}{6}i^2$$
$$\left(n-\frac{S_n}{R}\right)+\frac{n(n-1)}{2}i+\frac{n(n-1)(n-2)}{6}i^2=0$$
$$n(n-1)(n-2)i^2+3n(n-1)i+6\left(n-\frac{S_n}{R}\right)=0$$
$$n(n-1)(n-2)i^2+3n(n-1)i+6\left(n-\frac{S_n}{R}\right)=0$$
$$i^2+\frac{3}{(n-2)}i+\frac{6\left(n-\frac{S_n}{R}\right)}{n(n-1)(n-2)}=0$$

Solving the above quadratic equation for *i*, we get:

$$i=\frac{-\frac{3}{n-2}+\sqrt{\left(\frac{3}{n-2}\right)^2-4\frac{6\left(n-\frac{S_n}{R}\right)}{n(n-1)(n-2)}}}{2}$$

Simplifying the above equation, we get:

$$\therefore\ i=\frac{1}{2(n-2)}\left[\left(9-\frac{24(n-2)\left(n-\frac{S_n}{R}\right)}{n(n-1)}\right)^{\frac{1}{2}}-3\right] \quad ,\forall \quad \frac{S_n}{R}>n \qquad (9)$$

Therefore, equation(9) represents the annuity interest rate equation for computing *i* after the three ith term expansion.

b) When the Present Value is Known (A_n)

(1) Two_Term Simplification:

Using Eq.(4), we can get the following formulae:

$$\frac{A_n}{R}i = 1-(1+i)^{-n} \qquad (10)$$

From eq.(10) & the two ith term expansion of eq.(7), We have:

$$\frac{A_n}{R}i = 1-\left[1+(-n)\,i+\frac{[(-n)(-n+1)]}{2!}i^2\right]$$
$$=1-\left[1-n\,i+\frac{n(n-1)}{2}i^2\right]$$
$$=1-1+n\,i-\frac{n(n-1)}{2}i^2$$

$$= n\,i - \frac{n(n-1)}{2} i^2$$

Dividing both sides by *i*, we get:

$$\frac{A_n}{R} = n - \frac{n(n-1)}{2} i$$

$$2\left(n - \frac{A_n}{R}\right) = n(n-1)i$$

$$\therefore i = \frac{2\left(n - \frac{A_n}{R}\right)}{n(n-1)} \quad ,\forall \quad \frac{A_n}{R} < n \qquad (11)$$

Therefore, equation(11) represents the annuity interest rate equation for computing *i* after the two i^{th} term expansion.

(2) Three_Term Simplification

From eq.(10) & the three i^{th} term expansion of eq.(7), We have:

$$\frac{A_n}{R} i = 1 - (1+i)^{-n}$$

$$= 1 - \left[1 + (-n)\,i + \frac{(-n)(-n+1)}{2!} i^2 + \frac{(-n)(-n+1)(-n+2)}{3!} i^3\right]$$

$$= 1 - \left[1 - n\,i + \frac{n(n-1)}{2} i^2 - \frac{n(n-1)(n-2)}{6} i^3\right]$$

$$= n\,i - \frac{n(n-1)}{2} i^2 + \frac{n(n-1)(n-2)}{6} i^3$$

Dividing both sides by *i*, we get:

$$\frac{A_n}{R} = n - \frac{n(n-1)}{2} i + \frac{n(n-1)(n-2)}{6} i^2$$

$$\left(n - \frac{A_n}{R}\right) - \frac{n(n-1)}{2} i + \frac{n(n-1)(n-2)}{6} i^2 = 0$$

$$n(n-1)(n-2)i^2 - 3n(n-1)i + 6\left(n - \frac{A_n}{R}\right) = 0$$

$$i^2 - \frac{3}{(n-2)} i + \frac{6\left(n - \frac{A_n}{R}\right)}{n(n-1)(n-2)} = 0$$

Solving the above quadratic equation for *i*, we get:

$$i = \frac{\frac{3}{n-2} - \sqrt{\left(\frac{-3}{n-2}\right)^2 - 4\frac{6\left(n - \frac{A_n}{R}\right)}{n(n-1)(n-2)}}}{2}$$

Simplifying the above equation, we get:

$$\therefore i = \frac{1}{2(n-2)}\left[3 - \left(9 - \frac{24(n-2)\left(n - \frac{A_n}{R}\right)}{n(n-1)}\right)^{\frac{1}{2}}\right] \quad ,\forall \quad \frac{A_n}{R} < n \qquad (12)$$

Therefore, equation(12) represents the annuity interest rate equation for computing *i* after the three i^{th} term expansion.

5. CALCULATION OF ANNUITY INTEREST RATE

a) Tabulated Annuity Interest Rate

(1) Known Amount:

Table_1 includes selection of annuity interest rate used in the investment market. The ratio in Table_1 has been computed for given values of conversion period(n) and the corresponding annuity interest rate. This ratio is used back to extract the annuity interest rate(i_tabulated) from Tables given in [1].

i% exact	Time period (n)			
	n=10		n=20	
	i_tabulated	S_n/R	i_tabulated	S_n/R
0.2	0.249	10.0904816840387	0.248	20.3845990093093
0.4	0.416	10.1819335047275	0.416	20.7785540890338
0.6	0.624	10.2743656882306	0.584	21.1821069182341
0.8	0.872	10.3677885591048	0.868	21.5955054350601
1	1.000	10.4622125411205	1.000	22.0190039947967
1.2	1.247	10.5576481580867	1.133	22.4528635317327
1.4	1.493	10.6541060346834	1.268	22.8973517249426
1.6	1.623	10.7515968972984	1.515	23.3527431680687
1.8	1.868	10.8501315748704	1.758	23.8193195431968
2	2.000	10.9497209997379	2.000	24.2973697989177
2.2	2.244	11.0503762084931	2.238	24.7871903326693
2.4	2.488	11.1521083428429	2.475	25.2890851774580
2.6	2.511	11.2549286504744	2.525	25.8033661930578
2.8	2.756	11.3588484859271	2.764	26.3303532617892
3	3.000	11.4638793114707	3.000	26.8703744889805
3.2	3.242	11.5700326979890	3.233	27.4237664082190
3.4	3.483	11.6773203258690	3.464	27.9908741914986
3.6	3.516	11.7857539858976	3.536	28.5720518643747
3.8	3.758	11.8953455801620	3.769	29.1676625262402
4	4.000	12.0061071229586	4.000	29.7780785758355
4.2	4.037	12.1180507417060	4.084	30.4036819421117
4.4	4.479	12.2311886778653	4.453	31.0448643205664
4.6	4.520	12.3455332878658	4.547	31.7020274151745
4.8	4.953	12.4610970440374	4.895	32.3755831860388
5	5.000	12.5778925355488	5.000	33.0659541028884

TABLE 1: Computing the ratio S_n/R and tabulated annuity interest rate

(2) Known Present Value

Similarly, Table_2 computes the ratio for given values of conversion period(n) and the corresponding annuity interest rate. This ratio is used back to extract the annuity interest rate(i_tabulated) from Tables given in [1].

i% exact	Time period (n)			
	n=10		n=20	
	i_tabulated	A_n/R	i_tabulated	A_n/R
0.2	0.251	9.89087431187258	0.251	19.5860898344387
0.4	0.332	9.78347474743335	0.331	19.1840839823320
0.6	0.626	9.67776811620015	0.627	18.7935810581347
0.8	0.748	9.57372195913692	0.746	18.4141947010670
1	1.000	9.47130453070169	1.000	18.0455529662705
1.2	1.253	9.37048478137687	1.256	17.6872977422976
1.4	1.373	9.27123234066807	1.372	17.3390841937310
1.6	1.764	9.17351750055746	1.776	17.0005802277864
1.8	1.745	9.07731119939899	1.742	16.6714659838048
2	2.000	8.98258500624224	2.000	16.3514333445971
2.2	2.256	8.88931110557294	2.261	16.0401854686493
2.4	2.513	8.79746228245785	2.524	15.7374363422453
2.6	2.487	8.70701190808258	2.477	15.4429103506104
2.8	2.743	8.61793392567109	2.737	15.1563418672197
3	3.000	8.53020283677584	3.000	14.8774748604555
3.2	3.258	8.44379368792813	3.265	14.6060625168388
3.4	3.518	8.35868205763838	3.532	14.3418668800934
3.6	3.778	8.27484404373630	3.801	14.0846585053389
3.8	4.040	8.19225625104152	4.072	13.8342161277393
4	4.000	8.11089577935504	4.000	13.5903263449677
4.2	3.961	8.03074021176281	3.930	13.3527833128750
4.4	4.522	7.95176760324237	4.539	13.1213884537818
4.6	4.478	7.87395646956413	4.461	12.8959501768371
4.8	5.049	7.79728577647907	5.086	12.6762836099142
5	5.000	7.72173492918482	5.000	12.4622103425400

TABLE 2: Computing the ratio A_n/R and tabulated annuity interest rate

b) Simplified Annuity Interest Rate

(1) Known Amount
Table_3 computes annuity interest rate derived in Eq.(9), Eq.(10), Eq.(11), and Eq.(12) respectively. These Calculations have been computed for given values of S_n/R (Table_3_a) or A_n/R (Table_3_b) in addition to different conversion period(n).

Figure_1 shows the tabulated annuity interest rate, the exact annuity interest and the simplified one against different annuity interest rate. This figure indicates that the simplified annuity interest rate moves smoothly without any abrupt change or fluctuations. On the other hand, the tabulated annuity interest rate moves irregularly along with different interest rate. This variation reverses a wide range of errors associated with the tabulated calculation of annuity interest rate.

i% exact	i_Tabulated	Mathematical formula		Absolute Relative Error(ARE)%		
		i_2_Term	i_3_Term	Tabulated	2_Term	3_Term
0.2	0.249	0.2010	0.20000	24.718	0.53	0.001
0.4	0.416	0.4042	0.40002	4.088	1.07	0.007
0.6	0.624	0.6097	0.60009	4.048	1.61	0.016
0.8	0.872	0.8173	0.8002	9.003	2.16	0.028
1	1.000	1.0271	1.0004	1.07e-08	2.71	0.044
1.2	1.247	1.2392	1.2007	3.930	3.26	0.064
1.4	1.493	1.4535	1.4012	6.656	3.82	0.086
1.6	1.623	1.6702	1.6017	1.446	4.38	0.112
1.8	1.868	1.8891	1.8025	3.810	4.95	0.140
2	2.000	2.1104	2.0034	0.00019	5.52	0.172
2.2	2.244	2.3341	2.2045	2.038	6.09	0.207
2.4	2.488	2.5602	2.4058	3.690	6.67	0.245
2.8	2.756	3.0196	2.8092	1.559	7.84	0.329
3	3.000	3.2530	3.0112	0.00069	8.43	0.375
3.2	3.242	3.4889	3.2135	1.328	9.03	0.424
3.4	3.483	3.7273	3.4161	2.467	9.62	0.475
3.6	3.516	3.9683	3.6190	2.326	10.23	0.529
3.8	3.758	4.2118	3.8222	1.086	10.83	0.586
4	4.000	4.4580	4.0258	0.000059	11.45	0.645
4.2	4.037	4.7067	4.2297	3.873	12.06	0.707
4.4	4.479	4.9581	4.4339	1.798	12.68	0.771
4.6	4.520	5.2122	4.6385	1.717	13.31	0.837
4.8	4.953	5.4691	4.8435	3.200	13.93	0.906
5	5.000	5.7286	5.0488	0.00073	14.57	0.977

TABLE 3: a)Annuity Interest Rate at known S_n/R and time period n=10

i% exact	i_Tabulated	Mathematical formula		Absolute Relative Error(ARE)%		
		i_2_Term	i_3_Term	Tabulated	2_Term	3_Term
0.2	0.248	0.20	0.20	24.40	1.21	0.01
0.4	0.416	0.40	0.40	4.00	2.44	0.04
0.6	0.584	0.62	0.60	2.62	3.69	0.09
0.8	0.868	0.83	0.80	8.58	4.96	0.15
1	1.000	1.06	1.00	0.00	6.26	0.24
1.2	1.133	1.29	1.20	5.56	7.58	0.33
1.4	1.268	1.52	1.41	9.39	8.92	0.45
1.6	1.515	1.76	1.60	5.32	10.28	0.58
1.8	1.758	2.01	1.81	2.29	11.67	0.72
2	2.000	2.26	2.017	0.00	13.08	0.88
2.2	2.238	2.51	2.22	1.76	14.52	1.05
2.4	2.475	2.78	2.43	3.12	15.98	1.23
2.8	2.764	3.33	2.84	1.29	18.99	1.64
3	3.000	3.61	3.05	0.00	20.53	1.86
3.2	3.233	3.90	3.26	1.05	22.10	2.09
3.4	3.464	4.20	3.47	1.89	23.69	2.33
3.6	3.536	4.51	3.69	1.77	25.32	2.59
3.8	3.769	4.82	3.90	0.80	26.97	2.86
4	4.000	5.14	4.12	0.00	28.65	3.13
4.2	4.084	5.47	4.34	2.76	30.37	3.42
4.4	4.453	5.81	4.56	1.21	32.11	3.72
4.6	4.547	6.15	4.78	1.14	33.89	4.02
4.8	4.895	6.51	5.01	1.99	35.69	4.34
5	5.000	6.87	5.23	0.00	37.53	4.67

TABLE 3: b)Annuity Interest Rate at known S_n/R and time period n=20

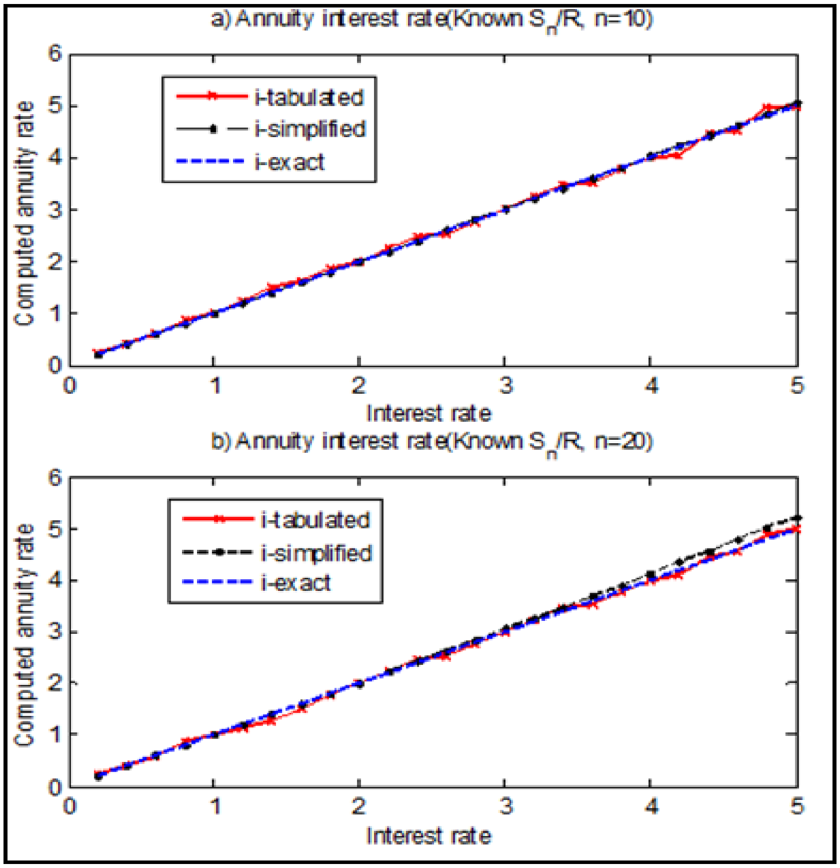

FIGURE 1: Tabulated and Simplified annuity interest rate (known S_n/R)

(2) Known Present Value:
The tabulated annuity interest rate and the simplified one are shown in Table_4. These Calculations have been computed for given values of A_n/R in addition to different conversion period(n).

Figure_2 shows the tabulated annuity interest rate, the exact annuity interest and the simplified one against different annuity interest rate for known values of A_n/R. This figure indicates that the simplified annuity interest rate moves smoothly without any abrupt change or fluctuations. On the other hand, the tabulated annuity interest rate moves irregularly along with different interest rate. This variation reverses a wide range of errors associated with the tabulated calculation of annuity interest rate.

i% exact	i_Tabulated	Mathematical formula		Absolute Relative Error(ARE)%		
		i_2_Term	i_3_Term	Tabulated	2_Term	3_Term
0.2	0.251	0.243	0.244	25.343	21.251	22.045
0.4	0.332	0.481	0.488	16.968	20.292	21.876
0.6	0.626	0.716	0.730	4.309	19.345	21.715
0.8	0.748	0.947	0.973	6.502	18.411	21.563
1	1.000	1.175	1.214	0.000	17.488	21.419
1.2	1.253	1.399	1.455	4.445	16.577	21.284
1.4	1.373	1.619	1.696	1.915	15.677	21.158
1.6	1.764	1.837	1.937	10.244	14.789	21.040
1.8	1.745	2.050	2.177	3.033	13.912	20.932
2	2.000	2.261	2.417	0.001	13.046	20.833
2.2	2.256	2.468	2.656	2.539	12.191	20.744
2.4	2.513	2.672	2.896	4.708	11.346	20.664
2.8	2.743	3.071	3.375	2.037	9.688	20.536
3	3.000	3.266	3.615	0.000	8.874	20.488
3.2	3.258	3.458	3.854	1.822	8.070	20.450
3.4	3.518	3.647	4.094	3.462	7.276	20.424
3.6	3.778	3.834	4.335	4.955	6.491	20.410
3.8	4.040	4.017	4.575	6.320	5.716	20.407
4	4.000	4.198	4.817	0.001	4.950	20.417
4.2	3.961	4.376	5.058	5.702	4.194	20.440
4.4	4.522	4.552	5.301	2.778	3.446	20.477
4.6	4.478	4.725	5.544	2.653	2.707	20.527
4.8	5.049	4.895	5.788	5.186	1.978	20.592
5	5.000	5.063	6.034	0.000	1.256	20.672

TABLE 4: a)Annuity Interest Rate at known $A_n/R$and time period n=10

i% exact	i_Tabulated	Mathematical formula		Absolute Relative Error(ARE)%		
		i_2_Term	i_3_Term	Tabulated	2_Term	3_Term
0.2	0.251	0.218	0.221	25.652	8.924	10.386
0.4	0.331	0.429	0.441	17.239	7.357	10.276
0.6	0.627	0.635	0.661	4.434	5.826	10.198
0.8	0.746	0.835	0.881	6.725	4.329	10.154
1	1.000	1.029	1.101	0.000	2.866	10.145
1.2	1.256	1.217	1.322	4.688	1.434	10.174
1.4	1.372	1.400	1.543	2.028	0.034	10.243
1.6	1.776	1.579	1.766	10.993	1.335	10.357
1.8	1.742	1.752	1.989	3.250	2.674	10.517
2	2.000	1.920	2.215	0.000	3.985	10.728
2.2	2.261	2.084	2.442	2.763	5.268	10.995
2.4	2.524	2.243	2.672	5.158	6.523	11.323
2.8	2.737	2.549	3.141	2.243	8.954	12.193
3	3.000	2.696	3.383	0.001	10.131	12.752
3.2	3.265	2.839	3.629	2.029	11.284	13.411
3.4	3.532	2.978	3.882	3.879	12.413	14.185
3.6	3.801	3.113	4.143	5.579	13.518	15.095
3.8	4.072	3.245	4.414	7.152	14.601	16.166
4	4.000	3.374	4.698	0.000	15.662	17.438
4.2	3.930	3.499	4.996	6.426	16.702	18.960
4.4	4.539	3.620	5.316	3.165	17.720	20.813
4.6	4.461	3.739	5.663	3.016	18.718	23.119
4.8	5.086	3.855	6.053	5.956	19.696	26.098
5	5.000	3.967	6.510	0.000	20.655	30.206

TABLE 4: b)Annuity Interest Rate at known $A_n/R$and time period n=20

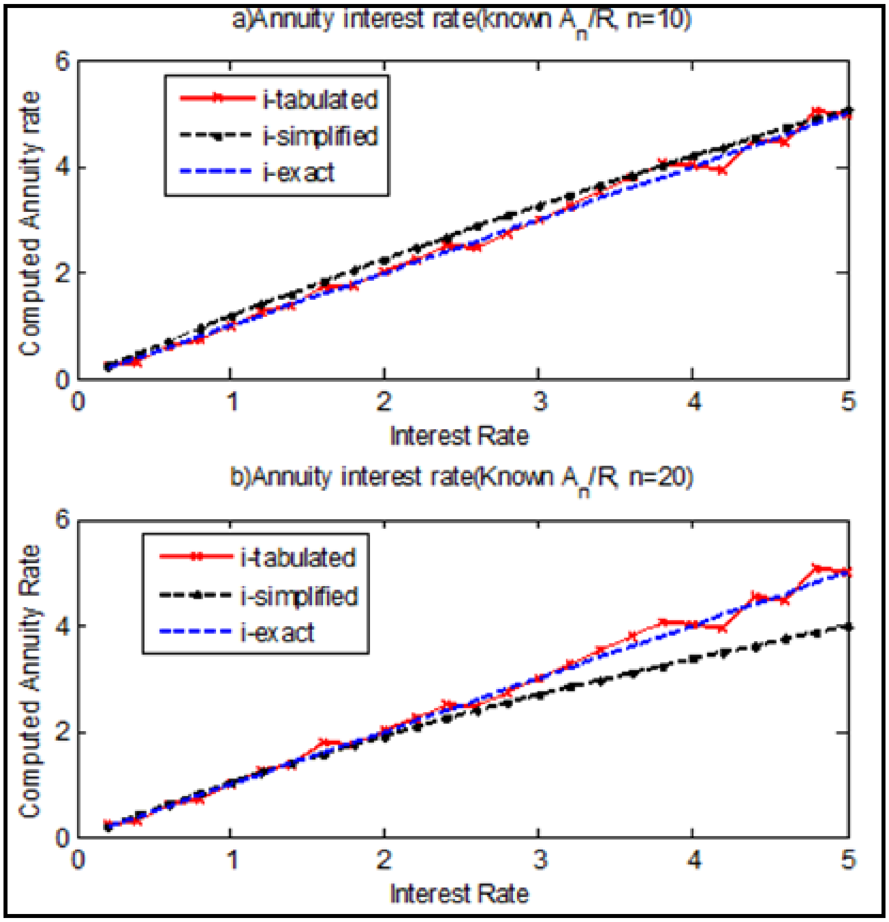

FIGURE 2: Tabulated and Simplified annuity interest rate (known A_n/R)

6.ERROR ANALYSIS OF ANNUITY INTEREST RATE

The percentage absolute relative error(ARE) between the exact and simplified Annuity Interest Rate is given by:

$$ARE = \left|\frac{exact - simplified}{exact}\right| * 100 \qquad (13)$$

Table_3 and Table_4 show the percentage Absolute Relative Error(ARE)of the tabulated and simplified annuity interest rate respectively. These tables compute the error associated with the annuity interest rate at known amount and present value respectively.Figure_3 shows the variation. Therefore, computing the annuity interest rateusing Eq.(9) ,three_Term_simplification, is recommended especially when theamount is known and lower conversion time period(n).

Similarly, Figure_4 indicates that the annuity interest rate using Eq.(11) ,two_Term_simplification, is recommended especially when the present value is known and lower conversion time period(n).

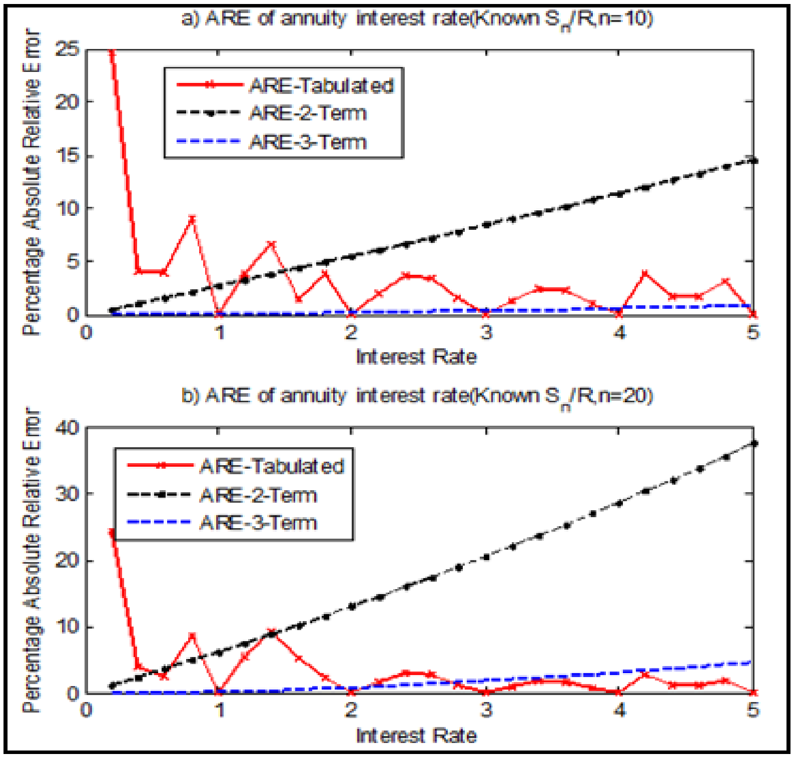

FIGURE 3: Percentage absolute relative error of annuity interest rate (known S_n/R)

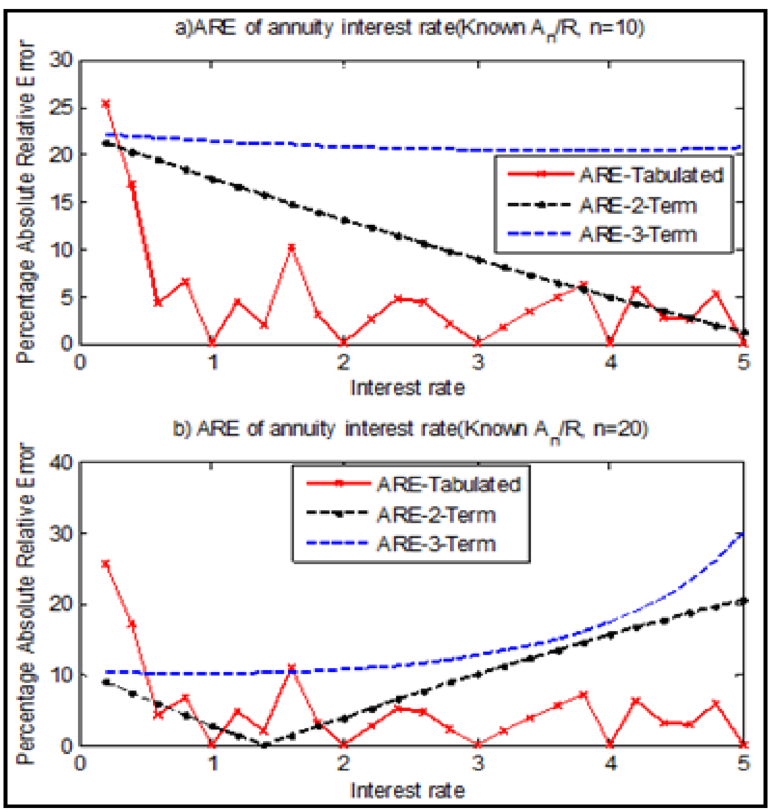

FIGURE 4: Percentage absolute relative error of annuity interest rate (known A_n/R)

The percentage reduction in Relative Error between the tabulated technique and the simplified one of the annuity interest rate is given by:

$$\delta E = \left(\frac{ARE\ of\ tabulated\ rate - ARE\ of simplified\ rate}{ARE\ tabulated\ rate}\right) * 100 \qquad (14)$$

Where:

δE isthe percentage reduction in Relative Error between the tabulated technique and the simplified one.If δE is positive values, Error reduction will occur using simplified technique. If it is negative values, Error reduction will occur using tabulated technique. otherwise, there is no error reduction.

Table_5 shows the percentage reduction in Relative Error between the tabulated technique and the simplified one for different interest rate.

The squared error(Er) between the exact values and the computed annuity interest rate is given by:

$$E_i = (exact - computed\ )^2$$
$$= (ARE * exact)^2 \qquad (15)$$

i% exact	Known S_n/R & n=10			Known A_n/R & n=10		
	Error reduction (%)	Squared error(E_i)		Error reduction (%)	Squared error(E_i)	
		Tabulated	3_Term		Tabulated	2_Term
0.2	99.99	24.44	1.4E-07	16.15	25.69	18.06
0.4	99.82	2.67	8.6E-06	-19.59	46.07	65.88
0.6	99.60	5.90	9.7E-05	-348.99	6.68	134.72
0.8	99.68	51.89	5.4E-04	-183.16	27.05	216.93
1.2	98.37	22.24	5.9E-03	-272.92	28.45	395.70
1.4	98.70	86.84	1.5E-02	-718.45	7.19	481.73
1.6	92.24	5.36	3.2E-02	-44.37	268.65	559.93
1.8	96.30	47.05	6.4E-02	-358.70	29.80	627.10
2.2	89.82	20.12	2.1E-01	-380.08	31.21	719.30
2.4	93.35	78.46	3.5E-01	-141.01	127.66	741.51
2.6	91.60	78.29	5.5E-01	-142.18	127.35	746.97
2.8	78.88	19.06	8.5E-01	-375.70	32.52	735.81
3.2	68.08	18.08	1.8E+00	-342.99	33.98	666.86
3.4	80.72	70.38	2.6E+00	-110.15	138.56	611.94
3.6	77.23	70.15	3.6E+00	-31.00	318.18	546.06
3.8	46.04	17.05	5.0E+00	9.56	576.82	471.79
4.2	81.74	264.72	8.8E+00	26.45	573.52	310.23
4.4	57.11	62.60	1.2E+01	-24.06	149.37	229.91
4.6	51.22	62.42	1.5E+01	-2.05	148.92	155.11
4.8	71.67	235.95	1.9E+01	61.87	619.73	90.10
Mean	83.61 %	______	______	_____	______	_______

TABLE 5: Squared error and Error reduction for computing Annuity Interest Rate

If the integer values of annuity interest rate in Table_5(integer values of i_exact) is excluded due to very small errors associated with it. This indicates that the 3_Term simplified technique of known amount gives reduction in ARE by approximately 83.61% compared to the tabulated technique. Figure_5.a shows the variation. On the other hand, the tabulated technique of known present value is appropriate compared to the simplified one. This led us to implement a correction factor which will be discussed in the coming paper.Figure_5.b shows the variation.

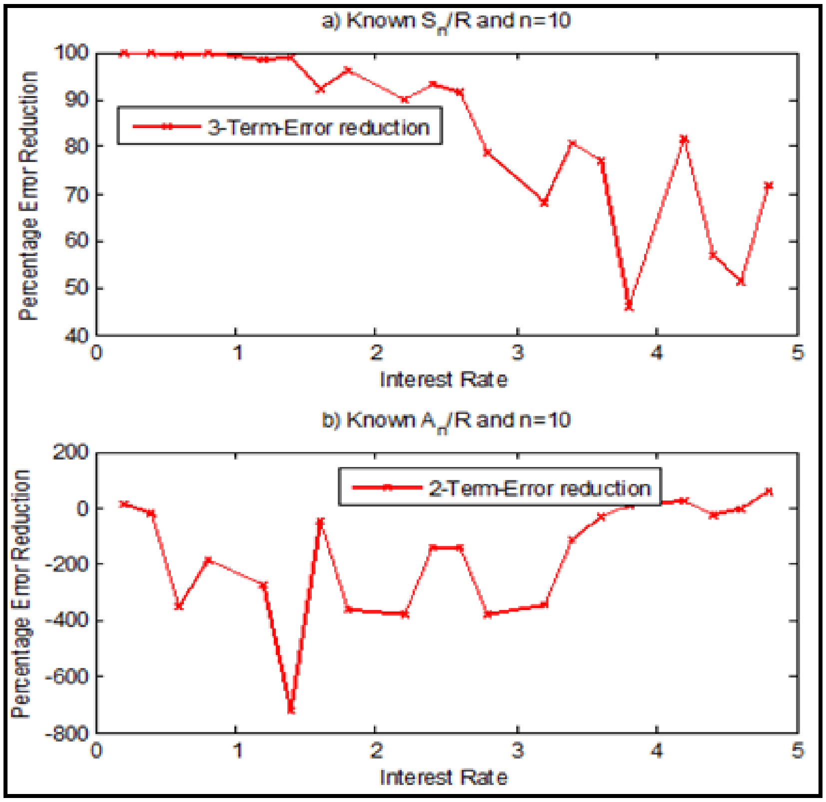

FIGURE 5: Percentage error reduction of annuity interest rate.

7. PROCESSING TIME OF THE ANNUITY INTEREST RATE

The processing time required for Computing the annuity interest rateis executed by LaptopDELL-inspiron-1520.Table_6 indicates that the average processing time required for computing the annuity interest rateusing the tabulated and the simplified technique.

i% exact	Known S_n/R & n=10		Known A_n/R & n=10	
	CPU time (Second)		CPU time (Second)	
	Tabulated	3_Term	Tabulated	2_Term
1.6	63.5	2.6137e-007	65	5.4142e-007

TABLE 6: Average CPU time of Annuity Interest Rate

8. MATLAB PROGRAMMING:

A complete program can be obtained by writing directly to the author[2].

9. COMPUTATIONAL ENERGY OF THE ANNUITY INTEREST RATE

Computing the computational energy for annuity interest rate requires the determination of conversion period(n), the square error(Ei), and the average processing time(CPU time). Therefore, consider the conversion period(n) represents the resistance, the square error is measured in [volts]2, and the CPU time in second. Then, the computational energy per conversion period is given by:

$$CE = \frac{E_i * t}{n} \qquad (16)$$

Where:

CE is the computational energy per conversion period.
$E_i$is the ith square error.
t isthe averageCPU time.

n is the conversion period.

The computational energy saved by the simplified technique compared to the tabulated one is given by:

$$\delta CE = \frac{CE_T - CE_s}{CE_T} * 100 \qquad (17)$$

Where:

δCE is the relative computational energy saved by the simplified technique.
CE_T is the computational energy for the Tabulated method.
CE_s is the computational energy for the simplified technique.

Table_7 shows the computational energy(CE) for each technique. This table indicates that the simplified technique saved computational energy by approximately 99.9% compared to the tabulated one. Figure(6) shows the variation in computational energy required for calculation.

i_exact %	Known S_n/R & n=10		
	CE_Tabulated	CE_3_Term	CE_saved by 3_Term
0.2	155.20	3.58E-15	100
0.4	16.98	2.26E-13	99.9999999999987
0.6	37.48	2.53E-12	99.9999999999933
0.8	329.47	1.40E-11	99.9999999999958
1.2	141.24	1.55E-10	99.9999999998906
1.4	551.42	3.84E-10	99.9999999999304
1.6	34.03	8.42E-10	99.9999999975242
1.8	298.74	1.68E-09	99.9999999994367
2.2	127.76	5.45E-09	99.9999999957343
2.4	498.21	9.06E-09	99.9999999981822
2.6	497.11	1.44E-08	99.9999999970962
2.8	121.01	2.22E-08	99.9999999816483
3.2	114.84	4.82E-08	99.9999999580682
3.4	446.94	6.84E-08	99.9999999847050
3.6	445.43	9.51E-08	99.9999999786590
3.8	108.29	1.30E-07	99.9999998801563
4.2	1680.97	2.31E-07	99.9999999862822
4.4	397.53	3.01E-07	99.9999999242805
4.6	396.35	3.88E-07	99.9999999020713
4.8	1498.30	4.95E-07	99.9999999669690
Mean	------------	----------	**99.99**

TABLE 7: Computational Energy of Annuity Interest Rate(CE)

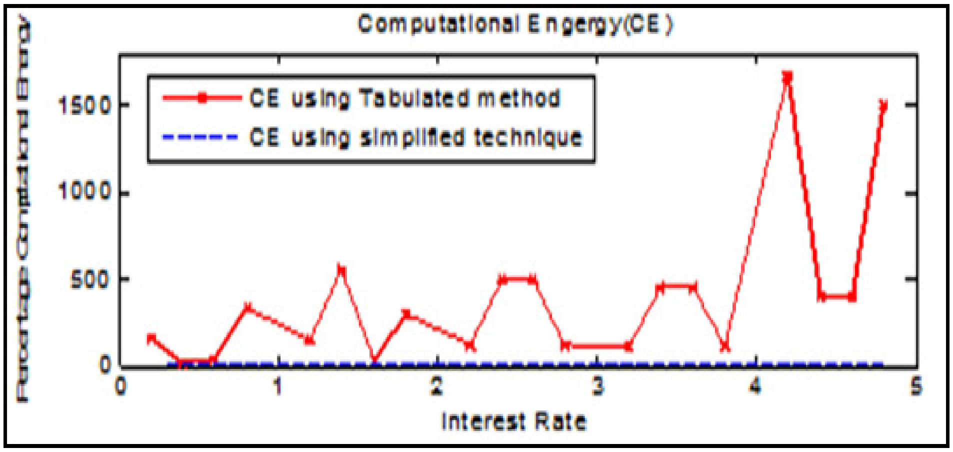

FIGURE 6: Percentage Computational Energy of annuity interest rate.

10. CONCLUSIONS

A new mathematical formula is derived here to compute an approximate value of the Annuity Interest Rate. It can be implemented using simple calculator, to save time and to avoid systematic errors associated with tables, and to calculate missing values of i in tables. Since other solutions depend on a trial-and-error approach.

A new algorithm has been derived for fast evaluation of the annuity interest rate. As a result the new technique offered four advantages over the tabulated one:

(1) It drastically reduces the average CPU time required for calculating the annuity interest rate.
(2) It drastically reduces the absolute relative error(ARE) for calculating the annuity interest rateby 83.61%compared to the current one.
(3) It gives minimum square error compared to the current tabulated method.
(4) It has lowest computational energy.

The aforementioned features are combined in a mathematical formula to describe the system performance. This formula is called the computational energy. A quantitative study has been carried out to compute the computational energy for each technique. The results show that the simplified technique saved computational energy by 99.9% compared to the current one. A correction factor will be discussed in the next paper.

11. REFERENCES

[1] Shao and Shao, “Mathematics for management and finance”, eighth edition, 1998.

[2] Email: karamfayed_1@hotmail.com