Designing SDRE-Based Controller for a Class of Nonlinear Singularly Perturbed Systems

Seyed Mostafa Ghadami *m.gadami@srbiau.ac.ir*
Department of Electrical Engineering,
Science and Research Branch,
Islamic Azad University, Tehran, Iran

Roya Amjadifard *amjadifard@tmu.ac.ir*
Assistant Professor,
Kharazmi University,
Tehran, Iran

Hamid Khaloozadeh *h_khaloozadeh@kntu.ac.ir*
Professor,
K.N. Toosi University of Technology,
Tehran, Iran

Abstract

Designing a controller for nonlinear systems is difficult to be applied. Thus, it is usually based on a linearization around their equilibrium points. The state dependent Riccati equation control approach is an optimization method that has the simplicity of the classical linear quadratic control method. On the other hand, the singular perturbation theory is used for the decomposition of a high-order system into two lower-order systems. In this study, the finite-horizon optimization of a class of nonlinear singularly perturbed systems based on the singular perturbation theory and the state dependent Riccati equation technique together is addressed. In the proposed method, first, the Hamiltonian equations are described as a state-dependent Hamiltonian matrix, from which, the reduced-order subsystems are obtained. Then, these subsystems are converted into outer-layer, initial layer correction and final layer correction equations, from which, the separated state dependent Riccati equations are derived. The optimal control law is, then, obtained by computing the Riccati matrices.

Keywords: Singularly Perturbed Systems, State-Dependent Riccati Equation, Nonlinear Optimal Control, Finite-Horizon Optimization Problem, Single Link Flexible Joint Robot Manipulator.

1. INTRODUCTION

Designing regulator systems is an important class of optimal control problems in which optimal control law leads to the Hamilton-Jacobi-Belman (HJB) equation. Various techniques have been suggested to solve this equation. One of these techniques, which are used for optimizing in infinite horizon, is based on the state-dependent Riccati equation (SDRE). In this technique, unlike linearization methods, a description of the system as state-dependent coefficients (SDCs) and in the form $f(x)=A(x)x$ must be provided. In this representation, $A(x)$ is not unique. Therefore, the solutions of the SDRE would be dependent on the choice of matrix $A(x)$. With suitable choice of the matrix, the solution to the equation is optimal; otherwise, the equation has suboptimal solutions. Bank and Mhana [1] proposed a suitable method for the selection of SDCs. Çimen [2] provided the condition for the solvability and local asymptotic stability of the SDRE closed-loop system for a class of nonlinear systems. Khaloozadeh and Abdolahi converted the nonlinear regulation [3] and tracking [4] problems in the finite-horizon to a state-dependent quasi-Riccati equation. They also provided an iterative method based on the Piccard theorem, which obtains a solution at a low convergence rate but good precision. On the other hand, the system discussed in this study is a class of nonlinear singularly perturbed systems. Naidu and Calise [5] dealt with the use of the singular perturbation theory and the two time scale (TTS) method in satellite and interplanetary trajectories, missiles, launch vehicles and hypersonic flight, space robotics. For LTI singularly perturbed systems, Su et al. [6] and Gajic et al. [7] performed the exact slow-fast decomposition of the linear quadratic (LQ) singularly perturbed optimal control problem in infinite horizon by deriving separate Riccati equations. Also, Gajic et al. [8] did the same for the case of finite horizon. Amjadifard et al. [9, 10] addressed the robust disturbance attenuation of a class of nonlinear singularly perturbed systems and robust regulation of a class of nonlinear singularly perturbed systems [11], and also position and velocity control of a flexible joint robot manipulator via fuzzy controller based on singular perturbation analysis [12]. Fridman [13, 14] dealt with the infinite horizon nonlinear quadratic optimal control problem for a class of non-standard nonlinear singularly perturbed systems by invariant manifolds of the Hamiltonian system and its decomposition into linear-algebraic Riccati equations.

In this study, we extend results of [13, 14] to the finite horizon by slow-fast manifolds of the Hamiltonian system and its decomposition into SDREs. Our contribution is that, we used the singular perturbation theory and SDRE method together. In the proposed method, first, the state-dependent Hamiltonian matrix is derived for the system under study. Then, this matrix is separated into the reduced-order slow and fast subsystems. Using the singular perturbation theory, the state equations and SDREs are converted into outer layer, initial layer correction and final layer correction equations, which are then solved to obtain the optimal control law. The block diagram of the proposed method is shown in Figure 1.

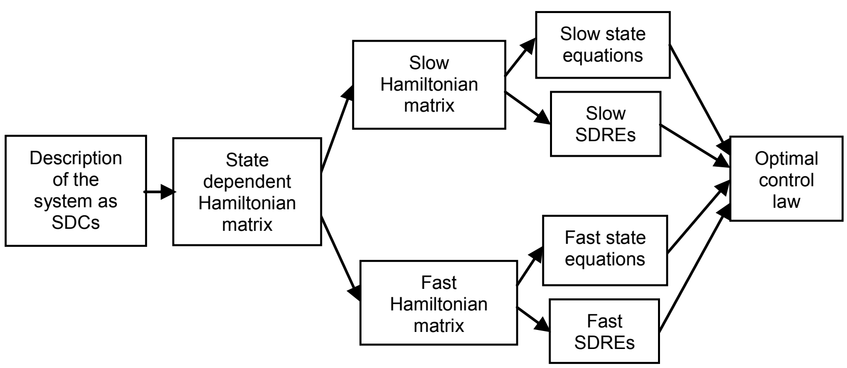

FIGURE 1: The design procedure stages in the proposed method.

The remainder of this study is organized as follows. Section 2 explains the structure of the singularly perturbed system for optimization. Section 3 involves in the description of steps of the design procedure in the proposed method. Section 4 presents the simulation results of the system used in the proposed method. Finally, the study culminates with indication of remarks in section 5.

2. PROBLEM FORMULATION

The following nonlinear singularly perturbed system is assumed:

$$E\dot{x} = f(x) + B(x)u, x(t_0) = x_0, \tag{1}$$

where $x(t) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, x_i \in R^{n_i}, i = 1,2$ are the states of system, and $x=0_n$ is the equilibrium point of the system ($n=n_1+n_2$). This system is full state observable, autonomous, nonlinear in the states, and affine in the input. Moreover, $f(x) = \begin{bmatrix} f_1(x_1,x_2) \\ f_2(x_1,x_2) \end{bmatrix}, f_i \in R^{n_i}, B(x) = \begin{bmatrix} B_1(x_1,x_2) \\ B_2(x_1,x_2) \end{bmatrix}, B_i \in R^{n_i}, i = 1,2$ are differentiable with respect to x_1, x_2 for a sufficient number of times. Furthermore, $f(0_n)=0_n$,

$B(x)\neq 0_{n\times m}$, $\forall x\in R^n$ and $E=\begin{bmatrix} I_{n_1\times n_1} & 0_{n_1\times n_2} \\ 0_{n_2\times n_1} & \varepsilon I_{n_2\times n_2}\end{bmatrix}$ that $\varepsilon>0$ is a small parameter. Provided these, it is desired to obtain the optimal control law $u(x)\in R^m$ such that for $k(x)\in R^n$, $k(0_n)=0_n$ and pointwise positive definite matrix $R(x)\in R^n\to R^{m\times m}$, the following performance index $\mathcal{J}$ is minimized.

$$\mathcal{J}=h(x(t_F))+\int_{t_0}^{t_F}\left(k^T(x)k(x)+u^T R(x)u\right)dt \tag{2}$$

Suppose that $k(x)$, $R(x)$ are differentiable with respect to x_1, x_2 for a sufficient number of times. Moreover, t_F is chosen such that it is sufficiently large with respect to the dominant time constant of the slow subsystem, and $x(t_F)$ is free.

3. THE PROPOSED METHOD

The singularly perturbed system (1) with performance index (2) is assumed. Defining the co-state vector $\lambda(x)=\begin{bmatrix}\lambda_1(x_1,x_2)\\ \lambda_2(x_1,x_2)\end{bmatrix}, \lambda_i\in R^{n_i}, i=1,2,$ the Hamiltonian function is obtained as (3):

$$\mathrm{H}(x,u,\lambda)=\frac{1}{2}k^T(x)k(x)+\frac{1}{2}u^T R(x)u+\lambda_1^{\,T}(f_1(x_1,x_2)+B_1(x_1,x_2)u)+\lambda_2^{\,T}(f_2(x_1,x_2)+B_2(x_1,x_2)u). \tag{3}$$

According to the optimal control theory, the necessary conditions for optimization would be as follow [2]:

$$\dot{x}_1=(\frac{\partial H}{\partial \lambda_1})^T=f_1(x_1,x_2)+B_1(x_1,x_2)u,\quad x_1(t_0), \tag{4a}$$

$$\varepsilon\dot{x}_2=(\frac{\partial H}{\partial \lambda_2})^T=f_2(x_1,x_2)+B_2(x_1,x_2)u,\quad x_2(t_0), \tag{4b}$$

$$\dot{\lambda}_1=-\left(\frac{\partial H}{\partial x_1}\right)^T=-\left(\frac{\partial k(x)}{\partial x_1}\right)^T k(x)-\left(\left(\frac{\partial f(x)}{\partial x_1}\right)^T+u^T\left(\frac{\partial B(x)}{\partial x_1}\right)^T+\frac{1}{2}u^T\frac{\partial R(x)}{\partial x_1}\right)\lambda$$
$$,\lambda_1(x(t_F))=\frac{1}{2}\left(\frac{\partial h}{\partial x_1}\right)^T|_{t_F}, \tag{4c}$$

$$\varepsilon\dot{\lambda}_2=-\left(\frac{\partial H}{\partial x_2}\right)^T=-\left(\frac{\partial k(x)}{\partial x_2}\right)^T k(x)-\left(\left(\frac{\partial f(x)}{\partial x_2}\right)^T+u^T\left(\frac{\partial B(x)}{\partial x_2}\right)^T+\frac{1}{2}u^T\frac{\partial R(x)}{\partial x_2}\right)\lambda$$
$$,\varepsilon\lambda_2(x(t_F))=\frac{1}{2}\left(\frac{\partial h}{\partial x_2}\right)^T|_{t_F}, \tag{4d}$$

$$0=\frac{\partial H}{\partial u}=R(x)u+B_1^{\,T}(x_1,x_2)\lambda_1+B_2^{\,T}(x_1,x_2)\lambda_2. \tag{4e}$$

3.1 Description of The System As SDCs (The first step)

A continuous nonlinear matrix-valued function $A(x)$ always exists such that

$$f(x)=A(x)x \tag{5}$$

Where $A(x):R^n\to R^{n\times n}$ is found by mathematical factorization and is, clearly, non-unique when n>1. A suitable choice for matrix $A(x)$ is $A(x)=\int_0^1\frac{\partial f}{\partial x}_{|x=\alpha x}d\alpha,$ where α is a dummy variable that was introduced in the integration [1]. Then, the relations (4) can be written as:

$$E\dot{x} = A(x)x + B(x)u, \quad x(t_0) \tag{6a}$$

$$E\dot{\lambda} = -(\frac{\partial H}{\partial x})^T = -\left(\frac{\partial k(x)}{\partial x}\right)^T k(x) - \left(\left(\frac{\partial f(x)}{\partial x}\right)^T + u^T\left(\frac{\partial B(x)}{\partial x}\right)^T + \frac{1}{2}u^T\frac{\partial R(x)}{\partial x}\right)\lambda, E\lambda(x(t_f)) = \frac{1}{2}\left(\frac{\partial h}{\partial x}\right)^T |_{t_f} \tag{6b}$$

$$u = -R^{-1}(x)B^T(x)\lambda \tag{6c}$$

Considering that *B(x)* and *R(x)* are nonzero, the optimal control law is proportional to vector λ.

3.2 Description of The Hamiltonian Matrix As SDCs (The second step)

Assuming that $K(x) = \int_0^1 \frac{\partial k}{\partial x}_{|x=\alpha x} d\alpha$ is available from *k(x)=K(x)x* and that $Q(x)=K^T(x)K(x)$ and $S(x)=B(x)R^{-1}(x)B^T(x)$, the relations (6) can be rewritten as follow:

$$E\dot{x} = A(x)x - S(x)\lambda, \quad x(t_0) = x_0, \tag{7a}$$

$$E\dot{\lambda} = -Q(x)x - A^T(x)\lambda - [\sum_{i=1}^{n} x_i\left(\frac{\partial K_i(x)}{\partial x}\right)^T k(x)$$
$$+\{\sum_{i=1}^{n} x_i\left(\frac{\partial A_i(x)}{\partial x}\right)^T + \frac{1}{2}\sum_{i=1}^{m}\left(\left(\lambda^T B(x)R^{-1}(x)\right)_i \frac{\partial R_i(x)}{\partial x}\right)R(x)^{-1}B^T(x) - \sum_{i=1}^{m}\left(\lambda^T B(x)R^{-1}(x)\right)_i\left(\frac{\partial B_i(x)}{\partial x}\right)^T\}\lambda], \tag{7b}$$
$$E\lambda(x(t_F)) = \frac{1}{2}\left(\frac{\partial h}{\partial x}\right)^T |_{t_F},$$

Where,

$$\frac{\partial A_i(x)}{\partial x} = \begin{bmatrix} \frac{\partial A_{1i}(x)}{\partial x_1} & \cdots & \frac{\partial A_{1i}(x)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial A_{ni}(x)}{\partial x_1} & \cdots & \frac{\partial A_{ni}(x)}{\partial x_n} \end{bmatrix}, \tag{8a}$$

$$\frac{\partial K_i(x)}{\partial x} = \begin{bmatrix} \frac{\partial K_{1i}(x)}{\partial x_1} & \cdots & \frac{\partial K_{1i}(x)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial K_{ni}(x)}{\partial x_1} & \cdots & \frac{\partial K_{ni}(x)}{\partial x_n} \end{bmatrix}, \tag{8b}$$

$$\frac{\partial B_i(x)}{\partial x} = \begin{bmatrix} \frac{\partial B_{1i}(x)}{\partial x_1} & \cdots & \frac{\partial B_{1i}(x)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial B_{ni}(x)}{\partial x_1} & \cdots & \frac{\partial B_{ni}(x)}{\partial x_n} \end{bmatrix}, \tag{8c}$$

$$\frac{\partial R_i(x)}{\partial x}=\begin{bmatrix}\frac{\partial R_{1i}(x)}{\partial x_1} & \cdots & \frac{\partial R_{1i}(x)}{\partial x_n}\\ \vdots & \ddots & \vdots\\ \frac{\partial R_{mi}(x)}{\partial x_1} & \cdots & \frac{\partial R_{mi}(x)}{\partial x_n}\end{bmatrix}. \tag{8d}$$

Assumption 1: A(x), B(x), Q(x), R(x), $\frac{\partial A(x)}{\partial x},\frac{\partial B(x)}{\partial x},\frac{\partial K(x)}{\partial x}$ and $\frac{\partial R(x)}{\partial x}$ are bounded in a neighborhood of Ω about the region. Then, the expression in the bracket will be ignored because of being small. This approximation is asymptotically optimal, in that it converges to the optimal control close to the origin as [2]. Thus, the relations (7) can be written as:

$$\begin{bmatrix}E\dot{x}\\ E\dot{\lambda}\end{bmatrix}=\begin{bmatrix}A(x) & -S(x)\\ -Q(x) & -A^T(x)\end{bmatrix}\begin{bmatrix}x\\ \lambda\end{bmatrix} \tag{9}$$

Remark 1: Suppose that T_{si}, T_{sF} are dominant time constants of the slow subsystem for initial and final layer correction, respectively. In other words, $T_{si}=\max\frac{1}{\left|real\left(eig_{slow}(J_i)\right)\right|}$ and $T_{sF}=\max\frac{1}{\left|real\left(eig_{slow}(J_F)\right)\right|}$ where, J_i and J_F are the Jacobian matrices of Hamiltonian system in initial and final layer correction and, $J_i\cong\begin{bmatrix}A(x) & -S(x)\\ -Q(x) & -A^T(x)\end{bmatrix}_{\substack{t=t_0\\ x=x_0\\ \lambda\to 0_n}}, J_F\cong\begin{bmatrix}A(x) & -S(x)\\ -Q(x) & -A^T(x)\end{bmatrix}_{\substack{t=t_F\\ x\to 0_n\\ \lambda\to 0_n}}$.

Note that $(T_{si}+T_{sF})/2$ is the average time constant of the Hamiltonian system and the setting time is fourfold of one, then a proper selection for t_F is

$$t_F > t_0+2(T_{si}+T_{sF}) \tag{10}$$

3.3 The Singularly Perturbed SDRE in Finite Horizon

In the proposed method, co-sate vector λ, can be described as $\lambda=P(x)x$ using the sweep method [3], where, $P(x)=\begin{bmatrix}P_{11}(x_1,x_2) & \varepsilon P_{21}^T(x_1,x_2)\\ P_{21}(x_1,x_2) & P_{22}(x_1,x_2)\end{bmatrix}, P_{ij}\in R^{n_i\times n_j}$ [7] is the unique, non-symmetric, positive-definite solution of the Riccati matrix equation. By differentiating λ with respect to time, we can write:

$$\dot{\lambda}=P(x)\dot{x}+\dot{P}(x)x \tag{11}$$

By substituting (11) in (9) and with rearrangement of one, we have:

$$E\dot{P}(x)+P^T(x)A(x)+A^T(x)P(x)-P^T(x)S(x)P(x)+Q(x)=0_{n\times n}, P\left(x(t_F)\right)=\frac{E^{-1}}{2}\int_0^1\frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x}\right)^T|_{x=\alpha x}\,d\alpha \tag{12}$$

The relation (12) is called a SDRE for nonlinear singularly perturbed system in finite horizon. It should be noted that the optimal control law is obtained by computing these Riccati matrices.
The solution conditions for SDRE are that $\{A(x),B(x)\}$ be stabilizable and $\{A(x),(Q(x))^{1/2}\}$ be detectable for $\forall x\in R^n$. A sufficient test for the stabilizability condition of $\{A(x),B(x)\}$ is to check that the controllability matrix $M_c=[B(x),A(x)B(x),\ldots,A^{n-1}(x)B(x)]$ has rank$(M_c)=n, \forall x\in\Omega$. Similarly, a sufficient test for detectability of $\{A(x), (Q(x))^{1/2}\}$ is that the observability matrix $M_o=[(Q(x))^{1/2}, (Q(x))^{1/2}A(x),\ldots, (Q(x))^{1/2}A^{n-1}(x)]$ has rank$(M_o)=n$, $\forall x\in\Omega$ [2]. Furthermore, the closed-loop matrix $A(x)-S(x)P(x)$ should be pointwise Hurwitz for $\forall x\in\Omega$. Here, Ω is any region such that the Lyapunov function $V(x)=x^T\left(\int_0^1\alpha P(\alpha x)d\alpha\right)x$ is locally Lipschitz around the origin [2]. The SDRE in

(12) consist $\frac{(n_1+n_2)(n_1+n_2+1)}{2}$ differential equations that number of these equations is reduced by using singular perturbation theory.

3.4 The Separated Hamiltonian Matrices

In the proposed method, by separating the slow and fast variables as $X_s=\begin{bmatrix} x_1 \\ \lambda_1(x_1,x_2) \end{bmatrix}$, $X_f=\begin{bmatrix} x_2 \\ \lambda_2(x_1,x_2) \end{bmatrix}$, we can describe the optimization relations (9) in the form of the following singularly perturbed state-dependent Hamiltonian matrix:

$$\begin{bmatrix} \dot{X}_s \\ \varepsilon\dot{X}_f \end{bmatrix}=\begin{bmatrix} H_{11}(x_1,x_2) & H_{12}(x_1,x_2) \\ H_{21}(x_1,x_2) & H_{22}(x_1,x_2) \end{bmatrix}\begin{bmatrix} X_s \\ X_f \end{bmatrix}, \tag{13}$$

Where, $H_{ij}(x_1,x_2)=\begin{bmatrix} A_{ij}(x_1,x_2) & -S_{ij}(x_1,x_2) \\ -Q_{ij}(x_1,x_2) & -(A_{ji}(x_1,x_2))^T \end{bmatrix}$ and *I, j=1,2*. Thus, we assume that the $2n_1$ eigenvalues of the system (13) are pointwise small and the remaining $2n_2$ eigenvalues are pointwise large, corresponding to the slow and fast responses, respectively. The state and co-state equations (13) constitute a singularly perturbed, two point boundary value problem (TPBVP). Hence, the asymptotic solution is obtained as an *outer* solution in terms of the original independent variable *t*, *initial* layer correction in terms of an initial stretched variable $\tau=\frac{t-t_0}{\varepsilon}$, and *final* layer correction in terms of a final stretched variable $\sigma=\frac{t_F-t}{\varepsilon}$ [5]. Thus, the composite solutions can be written as follow:

$$\begin{cases} x_1(t,\varepsilon)=x_{1o}(t,\varepsilon)+x_{1i}(\tau,\varepsilon)+x_{1F}(\sigma,\varepsilon) \\ x_2(t,\varepsilon)=x_{2o}(t,\varepsilon)+x_{2i}(\tau,\varepsilon)+x_{2F}(\sigma,\varepsilon) \\ P_s(t,\varepsilon)=P_{so}(t,\varepsilon)+P_{si}(\tau,\varepsilon)+P_{sF}(\sigma,\varepsilon) \\ P_f(t,\varepsilon)=P_{fo}(t,\varepsilon)+P_{fi}(\tau,\varepsilon)+P_{fF}(\sigma,\varepsilon) \end{cases} \tag{14}$$

where $t_0\le t\le t_F, 0\le\tau\le t_1\prec\frac{t_F-t_0}{\varepsilon}, 0\le\sigma\le t_2\prec\frac{t_F-t_0}{\varepsilon}$. The first terms on the right hand sides of the above relations represent the outer solution. The second and third terms represent boundary-layer corrections to the slow manifold near the initial and final times, respectively. Indices *o, i* and *F* correspond to the outer layer, initial, and final correction layers. For any boundary condition on the slow manifold, states and co-states are given by outer solution. For any boundary condition out of the slow manifold, the trajectory rapidly approaches the slow manifold according to the fast manifolds.

We now perform the slow-fast decomposition of the singularly perturbed state-dependent Hamiltonian matrix, in which $H_{22}(x_1,x_2)$ must be non-singular for all x_1, x_2 (in what follows, dependence upon x_1, x_2 is not represented, for convenience):

$$\begin{bmatrix} H_{11} & H_{12} \\ \frac{H_{21}}{\varepsilon} & \frac{H_{22}}{\varepsilon} \end{bmatrix}=\begin{bmatrix} I_{2n_1\times 2n_1} & \varepsilon H_{12}{H_{22}}^{-1} \\ -0_{2n_2\times 2n_2} & I_{2n_2\times 2n_2} \end{bmatrix}\times \begin{bmatrix} H_{11}-H_{12}{H_{22}}^{-1}H_{21} & 0_{2n_1\times 2n_2} \\ 0_{2n_2\times 2n_1} & \frac{H_{22}}{\varepsilon} \end{bmatrix}\times\begin{bmatrix} I_{2n_1\times 2n_1} & 0_{2n_1\times 2n_2} \\ {H_{22}}^{-1}H_{21} & I_{2n_2\times 2n_2} \end{bmatrix} \tag{15}$$

Stated differently:

$$\begin{bmatrix} I_{2n_1\times 2n_1} & -\varepsilon H_{12}H_{22}{}^{-1} \\ 0_{2n_2\times 2n_1} & I_{2n_2\times 2n_2} \end{bmatrix}\begin{bmatrix} \dot{X}_s \\ \dot{X}_f \end{bmatrix} = \begin{bmatrix} H_{11}-H_{12}H_{22}{}^{-1}H_{21} & 0_{2n_1\times 2n_2} \\ 0_{2n_2\times 2n_1} & \dfrac{H_{22}}{\varepsilon} \end{bmatrix} \times \begin{bmatrix} I_{2n_1\times 2n_1} & 0_{2n_1\times 2n_2} \\ H_{22}{}^{-1}H_{21} & I_{2n_2\times 2n_2} \end{bmatrix}\begin{bmatrix} X_s \\ X_f \end{bmatrix} \tag{16}$$

New co-sate vector can be described as $\lambda_{new}=P_{new}(x_{new})x_{new}$, where $x_{new}=\begin{bmatrix} x_s \\ x_f \end{bmatrix}$, $\lambda_{new}=\begin{bmatrix} \lambda_s(x_s,x_f) \\ \lambda_f(x_s,x_f) \end{bmatrix}$, $\lambda_s \in R^{n_1}, \lambda_f \in R^{n_2}$, and $P_{new}(x_{new})=\begin{bmatrix} P_s(x_s,x_f) & \varepsilon P_a(x_s,x_f) \\ \varepsilon P_b(x_s,x_f) & P_f(x_s,x_f) \end{bmatrix}$. Then, the new slow-fast variables are defined as follow:

$$\chi_s = \begin{bmatrix} x_s \\ \lambda_s(x_s,x_f) \end{bmatrix} = X_s, \tag{17a}$$

$$\chi_f = \begin{bmatrix} x_f \\ \lambda_f(x_s,x_f) \end{bmatrix} = H_{22}{}^{-1}H_{21}X_s + X_f, \tag{17b}$$

Thus, (13) is converted to a new form:

$$\begin{cases} \dot{X}_s - \varepsilon H_{12}H_{22}{}^{-1}\dot{X}_f = \left(H_{11}-H_{12}H_{22}{}^{-1}H_{21}\right)\chi_s \\ \varepsilon \dot{X}_f = H_{22}\chi_f \end{cases} \tag{18}$$

Finally, the optimization equations in a singular perturbation model framework with the new variables are obtained as:

$$\begin{cases} \dot{\chi}_s = \left(H_{11}-H_{12}H_{22}{}^{-1}H_{21}\right)\chi_s + H_{12}\chi_f \\ \varepsilon\dot{\chi}_f = \varepsilon H_{22}{}^{-1}H_{21}\left(H_{11}-H_{12}H_{22}{}^{-1}H_{21}\right)\chi_s + \varepsilon\left(-H_{22}{}^{-1}\dot{H}_{22}H_{22}{}^{-1}H_{21} + H_{22}{}^{-1}\dot{H}_{21}\right)\chi_s + \left(H_{22}+\varepsilon H_{22}{}^{-1}H_{21}H_{12}\right)\chi_f \end{cases} \tag{19}$$

Moreover, the separated state-dependent Hamiltonian matrices $H_s(x_s,x_f)$ and $H_{22}(x_1,x_2)$ are described in the form of the following:

$$\begin{aligned} H_s(x_s,x_f) &= H_{11}(x_1,x_2) - H_{12}(x_1,x_2)H_{22}{}^{-1}(x_1,x_2)H_{21}(x_1,x_2) + [O(\varepsilon)]_{2n_1\times 2n_1} \\ &= \begin{bmatrix} A_s(x_1,x_2)_{n_1\times n_1} & -S_s(x_1,x_2)_{n_1\times n_1} \\ -Q_s(x_1,x_2)_{n_1\times n_1} & -\left(A_s(x_1,x_2)\right)^T{}_{n_1\times n_1} \end{bmatrix} + [O(\varepsilon)]_{2n_1\times 2n_1}, \end{aligned} \tag{20a}$$

$$H_f(x_s,x_f) = H_{22}(x_1,x_2) + [O(\varepsilon)]_{2n_2\times 2n_2} = \begin{bmatrix} A_{22}(x_1,x_2)_{n_2\times n_2} & -S_{22}(x_1,x_2)_{n_2\times n_2} \\ -Q_{22}(x_1,x_2)_{n_2\times n_2} & -\left(A_{22}(x_1,x_2)\right)^T{}_{n_2\times n_2} \end{bmatrix} + [O(\varepsilon)]_{2n_2\times 2n_2}. \tag{20b}$$

3.5 The slow-fast SDREs (The third step)

In the proposed method, using the singular perturbation theory, the subsystems (19) are converted into outer-layer and boundary-layer correction subsystems. The separated SDRE relations are, then derived and solved for obtaining the optimal control law.

Theorem 1: The singularly perturbed system (1) with performance index (2) is assumed. The slow- fast state equations in the initial layer correction are obtained as follow:

$$\dot{x}_{1o} = \left(A_s(x_{1o}, x^*{}_{2o} + x_{2i}) - S_s(x_{1o}, x^*{}_{2o} + x_{2i})P_{so}\right)x_{1o}, x_{1o}|_{t_0} = x_1(t_0), \tag{21a}$$

$$\begin{aligned}\frac{dx_{2i}}{d\tau} &= \left(A_{22}(x_{1o}, x^*{}_{2o} + x_{2i}) - S_{22}(x_{1o}, x^*{}_{2o} + x_{2i})P^*{}_{22o}\right)\left(x^*{}_{2o} + x_{2i}\right) + \\ &\left(A_{21}(x_{1o}, x^*{}_{2o} + x_{2i}) - S_{21}(x_{1o}, x^*{}_{2o} + x_{2i})P_{so} - S_{22}(x_{1o}, x^*{}_{2o} + x_{2i})P^*{}_{21o}\right)x_{1o}, x_{2i}|_{t_0} = x_2(t_0) - x^*{}_{2o}(t_0),\end{aligned} \tag{21b}$$

Also, the slow- fast SDREs in the final layer correction are obtained as follow:

$$\begin{aligned}\dot{P}_{so} + P_{so}A_{so}(x_{1o}, x^*{}_{2o}) + A_{so}{}^T(x_{1o}, x^*{}_{2o})P_{so} - P_{so}S_{so}(x_{1o}, x^*{}_{2o})P_{so} + \\ Q_{so}(x_{1o}, x^*{}_{2o}) = 0_{n_1\times n_1}, P_{so}|_{t_F} = P_{11}(t_F),\end{aligned} \tag{22a}$$

$$\frac{dP_{fF}}{d\sigma} = P_{fF}\left(A_{22o} - S_{22o}P^*{}_{22o}\right) + \left(A_{22o}{}^T - P^*{}_{22o}S_{22o}\right)P_{fF} - P_{fF}S_{22o}P_{fF}, P_{fF}|_{t_F} = P_{22}(t_F) - P^*{}_{22o}(t_F). \tag{22b}$$

where, $\begin{bmatrix} P_{11}(x(t_F)) & \varepsilon P_{21}{}^T(x(t_F)) \\ P_{21}(x(t_F)) & P_{22}(x(t_F)) \end{bmatrix} = \frac{E^{-1}}{2}\int_0^1 \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x}\right)^T |_{x=\alpha x}\, d\alpha$. Furthermore, the optimal control law is as follows:

$$u = -R^{-1}(x_{1o}, x^*{}_{2o} + x_{2i})\left(B_1{}^T(x_{1o}, x^*{}_{2o} + x_{2i})P_{so}x_{1o} + B_2{}^T(x_{1o}, x^*{}_{2o} + x_{2i})\left(P_c x_{1o} + \left(P^*{}_{22o} + P_{fF}\right)\left(x^*{}_{2o} + x_{2i}\right)\right)\right), \tag{23}$$

where, P_{so} and P_{fF} are the unique, symmetric, positive-definite solutions of (22), and $P_c = \begin{bmatrix} P^*{}_{22o} + P_{fF} & -I_{n_2\times n_2} \end{bmatrix} H_{22}{}^{-1}(x_{1o}, x^*{}_{2o} + x_{2i})H_{21}(x_{1o}, x^*{}_{2o} + x_{2i})\begin{bmatrix} I_{n_1\times n_1} \\ P_{so} \end{bmatrix}$. The solution necessary conditions of relations (21) and (22) are as follow:

- $\{A_{so}(x_{1o},x^*{}_{2o}),\ B_{so}(x_{1o},x^*{}_{2o})\}$ and $\{A_{22o}(x_{1o},x^*{}_{2o}),\ B_{2o}(x_{1o},x^*{}_{2o})\}$ should be stabilizable for $\forall\left(x_{1o}, x^*{}_{2o}\right) \in R^{n_1} \times R^{n_2}$.
- $\{A_{so}(x_{1o},x^*{}_{2o}), (Q_{so}(x_{1o},x^*{}_{2o}))^{1/2}\}$ and $\{A_{22o}(x_{1o},x^*{}_{2o}),\ (Q_{22o}(x_{1o},x^*{}_{2o}))^{1/2}\}$ should be detectable for $\forall\left(x_{1o}, x^*{}_{2o}\right) \in R^{n_1} \times R^{n_2}$.
- The outer equations (24) should have solutions (the slow manifolds) as $x^*{}_{2o}(x_{1o},P_{11o})$, $P^*{}_{21o}(x_{1o},P_{11o})$ and $P^*{}_{22o}(x_{1o},P_{11o})$

$$\left(A_{21o} - S_{21o}P_{11o} - S_{22o}P_{21o}\right)x_{1o} + \left(A_{22o} - S_{22o}P_{22o}\right)x_{2o} = 0_{n_2}, \tag{24a}$$

$$\left(A_{22o}{}^T - P_{22o}S_{22o}\right)P_{21o} + \left(A_{12o}{}^T - P_{22o}S_{21o}\right)P_{11o} + P_{22o}A_{21o} + Q_{21o} = 0_{n_1\times n_2}, \tag{24b}$$

$$P_{22o}A_{22o} + A_{22o}{}^T P_{22o} - P_{22o}S_{22o}P_{22o} + Q_{22o} = 0_{n_2\times n_2}, \tag{24c}$$

It should be noted that in the above relations, all the elements of the state and Riccati matrices are dependent on state variables, and have not been represented for simplicity.□

Proofs of the theorems are given in appendix.

Remark 2: SDREs in (22) have $n_1 n_2$ the less differential equations respect to (12).

4. EXAMPLE

Consider a single link flexible joint robot manipulator as it has been introduced in [11]. This link is directly actuated by a D.C. electrical motor whose rotor is elastically coupled to the link. In this example, the mathematical model of system is as follows:

$$
\begin{aligned}
&I\ddot{q}_1 + mgl\sin(q_1) + k(q_1 - q_2) = 0 \\
&I\ddot{q}_2 + \beta\dot{q}_2 - k(q_1 - q_2) = u
\end{aligned} \tag{25}
$$

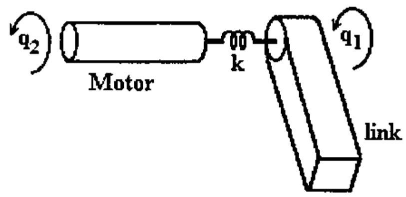

FIGURE 2: Single link flexible joint robot manipulator

In Table 1 there is a complete list of notations of the mathematical model of a single link flexible joint robot manipulator.

Notation	Description
q_1	angular positions of the link
q_2	angular positions of the motor
u	actuator force (motor torque)
I	the arm inertia
J	the motor inertia
β	the motor viscous friction
mgl	the nominal load in the rotor link
K	the stiffness coefficient of flexible joint

TABLE 1: Notations the mathematical model of a single link flexible joint robot manipulator.

Moreover, parameter values are given in Table 2.

parameter	Value of parameter
I	0.031(Kg.m^2)
J	0.004(Kg.m^2)
β	0.007
k	7.13
mgl	0.8 (N.m)

TABLE 2: Parameter values of the single link flexible joint robot manipulator.

Defining $x_1 = \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ \dot{q}_1 \end{bmatrix}, x_2 = \dot{q}_2, x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and ε=*J*, state equations are as follow:

$$
\begin{bmatrix} \dot{x}_{11} \\ \dot{x}_{12} \\ \dot{x}_{13} \\ \varepsilon\dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_{13} \\ x_2 \\ -\dfrac{mgl}{I}\sin(x_{11}) - \dfrac{k}{I}x_{11} + \dfrac{k}{I}x_{12} \\ kx_{11} - kx_{12} - \beta x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u, x_0 = \begin{bmatrix} 10^0 \\ 3^0 \\ 0^{0/s} \\ 0^{0/s} \end{bmatrix} \tag{26}
$$

It is desired to obtain the optimal control law such that the following performance index $\mathcal{J}$ is minimized.

$$\mathcal{J}=\int_0^5\left(x_{11}^2+x_{12}^2+x_{13}^2+x_2^2+u^2\right)dt \tag{27}$$

In this example, $f(x)=\begin{bmatrix} x_{13} \\ x_2 \\ -\frac{mgl}{I}\sin(x_{11})-\frac{k}{I}x_{11}+\frac{k}{I}x_{12} \\ kx_{11}-kx_{12}-\beta x_2 \end{bmatrix}, B(x)=\begin{bmatrix}0\\0\\0\\1\end{bmatrix}, k(x)=\begin{bmatrix}x_{11}\\x_{12}\\x_{13}\\x_2\end{bmatrix}, R(x)=1,$ and $h(x(t_F))=0$. Moreover, $f(x)$, $k(x)$ are differentiable with respect to x for a sufficient number of times and $x=0_4$ is the equilibrium point of the system. Furthermore, $t_0=0$, $t_F=5$, $P(x(t_F))=0_{4\times4}$.

Step 1 (*Description of the system as SDCs*):
To solve the optimization problem, the nonlinear functions $f(x)$, $k(x)$ must first be represented as SDCs. A suitable choice, considering [1], is as follows:

$$A(x)=\int_0^1 \frac{\partial f}{\partial x}_{|x=\alpha x}\,d\alpha=\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{mgl\sin(x_{11})}{Ix_{11}}-\frac{k}{I} & \frac{k}{I} & 0 & 0 \\ k & -k & 0 & -\beta \end{bmatrix} \tag{28a}$$

$$K(x)=\int_0^1 \frac{\partial k}{\partial x}_{|x=\alpha x}\,d\alpha=\begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{bmatrix} \tag{28b}$$

Step 2 (Description of the Hamiltonian matrix as SDCs):
The separated Hamiltonian matrices can be derived:

$$H_s(x_1,x_2)=\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{\beta k}{\beta^2+1} & -\frac{\beta k}{\beta^2+1} & 0 & 0 & -\frac{1}{\beta^2+1} & 0 \\ -\frac{mgl\sin(x_{11})}{Ix_{11}}-\frac{k}{I} & \frac{k}{I} & 0 & 0 & 0 & 0 \\ -1-\frac{k^2}{\beta^2+1} & \frac{k^2}{\beta^2+1} & 0 & 0 & -\frac{\beta k}{\beta^2+1} & \frac{mgl\sin(x_{11})}{Ix_{11}}+\frac{k}{I} \\ \frac{k^2}{\beta^2+1} & -1-\frac{k^2}{\beta^2+1} & 0 & 0 & \frac{\beta k}{\beta^2+1} & -\frac{k}{I} \\ 0 & 0 & -1 & -1 & 0 & 0 \end{bmatrix} \tag{29a}$$

$$H_{22}(x_1,x_2)=\begin{bmatrix} -\beta & -1 \\ -1 & -\beta \end{bmatrix} \tag{29b}$$

Step 3.1 (the outer equations):
The relations (24) have solutions as:

$$x^*_{2o}=\frac{\beta k(x_{11o}-x_{12o})-P_{so12}x_{11o}-P_{so22}x_{12o}-P_{so23}x_{13o}}{\beta^2+1} \tag{30a}$$

$$P^*{}_{21o} = \begin{bmatrix} k + \dfrac{P_{so12} - \beta k}{\sqrt{\beta^2+1}} \\ -k + \dfrac{P_{so22} + \beta k}{\sqrt{\beta^2+1}} \\ \dfrac{P_{so23}}{\sqrt{\beta^2+1}} \end{bmatrix} \tag{30b}$$

$$P^*{}_{22o} = \sqrt{\beta^2+1} - \beta \tag{30c}$$

Moreover, $\{A_{so}(x_{1o}, x^*{}_{2o}) = \begin{bmatrix} 0 & 0 & 1 \\ \dfrac{\beta k}{\beta^2+1} & -\dfrac{\beta k}{\beta^2+1} & 0 \\ -\dfrac{mgl\sin(x_{11o})}{Ix_{11o}} - \dfrac{k}{I} & \dfrac{k}{I} & 0 \end{bmatrix}, B_{so}(x_{1o}, x^*{}_{2o}) = \begin{bmatrix} 0 \\ \dfrac{1}{\sqrt{\beta^2+1}} \\ 0 \end{bmatrix}, \left(Q_{so}(x_{1o}, x^*{}_{2o})\right)^{\frac{1}{2}} =$

$\begin{bmatrix} \dfrac{1}{2} + \dfrac{\sqrt{\beta^2+1+2k^2}}{2\sqrt{\beta^2+1}} & \dfrac{1}{2} - \dfrac{\sqrt{\beta^2+1+2k^2}}{2\sqrt{\beta^2+1}} & 0 \\ \dfrac{1}{2} - \dfrac{\sqrt{\beta^2+1+2k^2}}{2\sqrt{\beta^2+1}} & \dfrac{1}{2} + \dfrac{\sqrt{\beta^2+1+2k^2}}{2\sqrt{\beta^2+1}} & 0 \\ 0 & 0 & 1 \end{bmatrix}\}$ is stabilizable and detectable. $\{A_{22o}(x_{1o}, x^*{}_{2o}) = -\beta,$

$B_{2o}(x_{1o}, x^*{}_{2o}) = 1, \left(Q_{22o}(x_{1o}, x^*{}_{2o})\right)^{\frac{1}{2}} = 1\}$ is also stabilizable and detectable.

Step 3.2 (the state equations):

According to (21), state variables relations in the initial layer correction are as follow:

$$\dot{x}_{1o} = \begin{bmatrix} x_{13o} \\ \dfrac{\beta k(x_{11o} - x_{12o}) - P_{so12}x_{11o} - P_{so22}x_{12o} - P_{so23}x_{13o}}{\beta^2+1} \\ -\dfrac{mgl}{I}\sin(x_{11o}) - \dfrac{k}{I}x_{11o} + \dfrac{k}{I}x_{12o} \end{bmatrix}, x_1(t_0) = \begin{bmatrix} 10^0 \\ 3^0 \\ 0^{0/s} \end{bmatrix} \tag{31a}$$

$$\frac{dx_{2i}}{d\tau} = -x_{2i}\sqrt{\beta^2+1}, x_{2i}(t_0) = \frac{7\beta k - 10P_{so12}(t_0) - 3P_{so22}(t_0)^0}{\beta^2+1} \tag{31b}$$

Step 3.3 (the slow-fast SDREs):

The slow- fast SDREs in (22) have *3* the less equations respect to the original SDRE. Considering (22), the SDRE relations in the final layer correction are as follow:

$$\dot{P}_{so} = \begin{bmatrix} \dot{P}_{s\_11o} & \dot{P}_{s\_12o} & \dot{P}_{s\_13o} \\ \left(\dot{P}_{s\_12o}\right)^T & \dot{P}_{s\_22o} & \dot{P}_{s\_23o} \\ \left(\dot{P}_{s\_13o}\right)^T & \left(\dot{P}_{s\_23o}\right)^T & \dot{P}_{s\_33o} \end{bmatrix}, P_{so}(t_F) = 0_{3\times3}$$

$$\begin{bmatrix} \dot{P}_{s\_11o} \\ \dot{P}_{s\_12o} \\ \dot{P}_{s\_13o} \\ \dot{P}_{s\_22o} \\ \dot{P}_{s\_23o} \\ \dot{P}_{s\_33o} \end{bmatrix} = \begin{bmatrix} \dfrac{2mglP_{sol3}\sin(x_{11o})}{Ix_{11o}} - 1 + \dfrac{2P_{sol3}k}{I} - \dfrac{2\beta kP_{sol2} + k^2 - P^2_{sol2}}{\beta^2+1} \\ \dfrac{mglP_{so23}\sin(x_{11o})}{Ix_{11o}} + k\dfrac{P_{so23} - P_{sol3}}{I} + \beta k\dfrac{P_{sol2} - P_{so2}}{\beta^2+1} + \dfrac{P_{sol2}P_{so22} + k^2}{\beta^2+1} \\ \dfrac{mglP_{so33}\sin(x_{11o})}{Ix_{11o}} + \dfrac{P_{sol2}P_{so23} - \beta kP_{so23}}{\beta^2+1} - P_{sol1} + k\dfrac{P_{so33}}{I} \\ -2k\dfrac{P_{so23}}{I} - 1 + \dfrac{2\beta kP_{so22} - k^2 + P^2_{so22}}{\beta^2+1} \\ \dfrac{P_{so22}P_{so23} + \beta kP_{so23}}{\beta^2+1} - P_{sol2} - k\dfrac{P_{so33}}{I} \\ \dfrac{P^2_{so23}}{\beta^2+1} - 1 - 2P_{sol3} \end{bmatrix} \tag{32a}$$

$$\frac{dP_{fF}}{d\sigma} = -2P_{fF}\sqrt{\beta^2+1} + P_{fF}{}^2,\ P_{fF}(t_F) = -\sqrt{\beta^2+1} + \beta \tag{32b}$$

Step 3.4 (the optimal control law):

Moreover, the optimal control law is as follow:

$$u = \frac{k}{\beta^2+1}(x_{11o} - x_{12o}) + \frac{\beta}{\beta^2+1}(P_{sol2}x_{11o} + P_{so22}x_{12o} + P_{so23}x_{13o}) - (P_{fF} + \sqrt{\beta^2+1} - \beta)x_{2i} \tag{33}$$

The state equations and SDREs are two-point boundary value problem (TPBVP) and dependent on state variables, but we have no state values in the whole interval [0,5]. To overcome this problem we solve the above equations by an iterative procedure [3, 4]. Now, running the simulation programs, Figures 3, 4 show the angular positions and velocities.

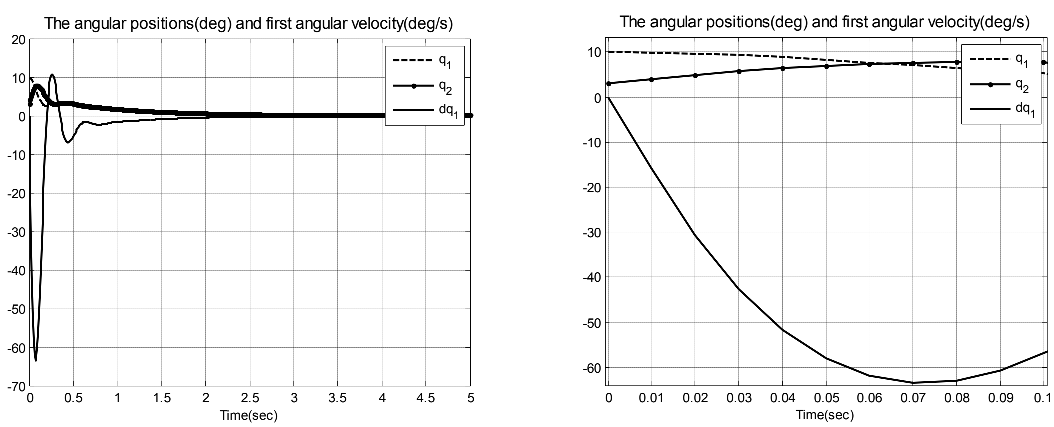

FIGURE 3: The slow state variables (The angular positions of q_1, q_2 and angular velocity of $\dot{q}_1$).

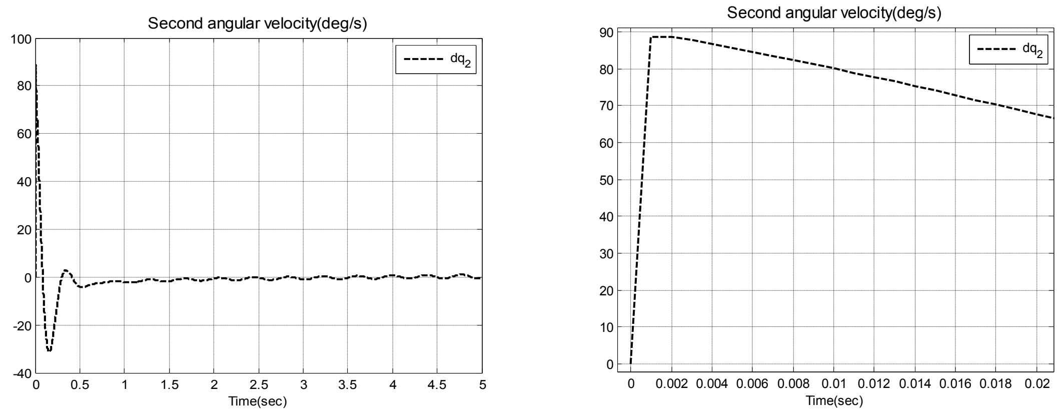

FIGURE 4: The fast state variable (angular velocity of $\dot{q}_2$).

Also, Figures 5 and 6 show the Riccati gains.

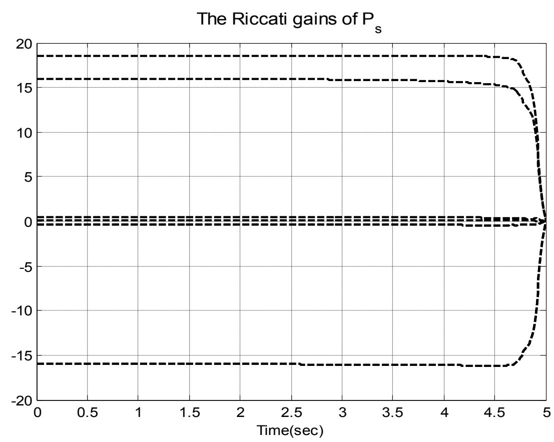

FIGURE 5: The Riccati gains of P_s.

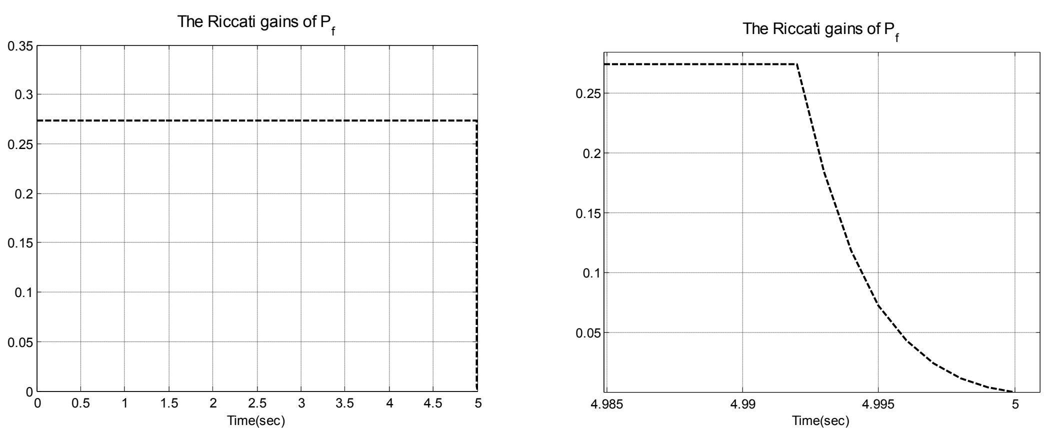

FIGURE 6: The Riccati gains of P_f.

From Figures 3 and 5, it can be seen that for any initial and final conditions on the slow manifold, for different values of ε , states are given by outer solution. On the other hand, Figures 4 and 6 show that for any initial and final conditions out of the slow manifold, the trajectories rapidly approach the slow manifold according to the fast manifolds. Moreover, Figure 7 shows the optimal control law.

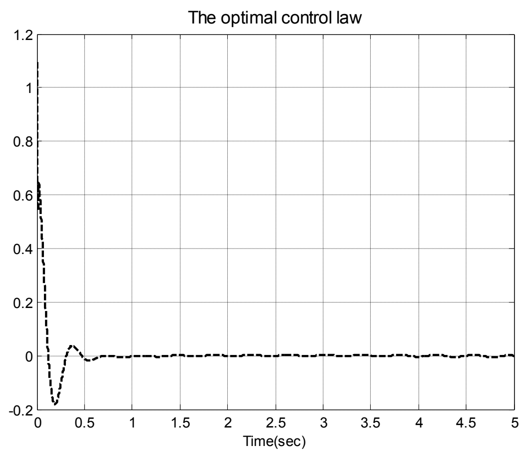

FIGURE 7: The optimal control law *u.*

5. CONCLUSION

With the proposed method in this study, it is seen that the finite-horizon optimization problem of a class of nonlinear singularly perturbed systems leads to SDREs for slow and fast state variables. One of the advantages of SDRE method is that knowledge of the Jacobian of the nonlinearity in the states, similar to HJB equation, is not necessary. Thus, the proposed method has not only simplicity of the LQ method but also higher flexibility, due to adjustable changes in the Riccati gains. On the other hand, one of the advantages of the singular perturbation theory is that it reduces high-order systems into two lower-order subsystems due to the interaction between slow and fast variables. Note that SDREs in the proposed method have n_1n_2 the less differential equations respect to the original SDRE. Thus, the slow-fast SDREs have the simpler computing than original SDRE and provide good approximations of one.

6. References

[1] S.P Banks and K.J. Mhana. “Optimal Control and Stabilization for Nonlinear Systems.” IMA Journal of Mathematical Control and Information, vol. 9, pp. 179-196, 1992.

[2] T. Çimen. ”State-Dependent Riccati Equation (SDRE) Control: A Survey,” in Proc. 17th World Congress; the International Federation of Automatic Control Seoul, Korea, 2008, pp. 3761-3775.

[3] H. Khaloozadeh and A. Abdolahi. “A New Iterative Procedure for Optimal Nonlinear Regulation Problem,” in Proc. III International Conference on System Identification and Control Problems, 2004, pp. 1256-1266.

[4] H. Khaloozadeh and A. Abdolahi. “An Iterative Procedure for Optimal Nonlinear Tracking Problem,” in Proc. Seventh International Conference on Control, Automation, Robotics and Vision, 2002, pp. 1508-1512.

[5] D.S. Naidu and A.J. Calise. “Singular Perturbations and Time Scales in Guidance and Control of Aerospace Systems: A Survey.” Journal of Guidance, Control and Dynamics, vol. 24, no.6, pp. 1057-1078, Nov.-Dec. 2001.

[6] W.C. Su, Z. Gajic and X. Shen. “The Exact Slow-Fast Decomposition of the Algebraic Riccati Equation of Singularly Perturbed Systems.” IEEE Transactions on Automatic Control, vol. 37, no. 9, pp. 1456-1459, Sep. 1992.

[7] Z. Gajic, X. Shen and M. Lim. ”High Accuracy Techniques for Singularly Perturbed Control Systems-an Overview.” PINSA, vol. 65, no. 2, pp. 117-127, March 1999.

[8] Z. Gajic, S. Koskie and C. Coumarbatch. “On the Singularly Perturbed Matrix Differential Riccati Equation,” in Proc. CDC-ECC'05, 44th IEEE Conference on Decision and Control and European Control Conference ECC 2005, Seville, Spain,2005, pp. 14–17.

[9] R. Amjadifard and M.T. H. Beheshti. "Robust disturbance attenuation of a class of nonlinear singularly perturbed systems." International Journal of Innovative Computing, Information and Control (IJICIC), vol. 6, pp. 1349-4198, 2010.

[10] R. Amjadifard and M.T.H Beheshti. "Robust stabilization for a class of nonlinear singularly perturbed systems." Journal of Dynamic Systems, Measurement and Control (ASME), In Press, 2011.

[11] R. Amjadifard, M.J. Yazdanpanah and M.T.H. Beheshti. "Robust regulation of a class of nonlinear singularly perturbed systems," IFAC, 2005.

[12] R. Amjadifard, S.E. Khadem and H. Khaloozadeh. "Position and velocity control of a flexible Joint robot manipulator via fuzzy controller based on singular perturbation analysis," IEEE International Fuzzy Systems Conference, 2001, pp. 348-351.

[13] E. Fridman. “Exact Slow-Fast Decomposition of the Nonlinear Singularly Perturbed Optimal Control Problem.” System and Control Letters, vol. 40, pp. 121-131, Jun. 2000.

[14] E. Fridman. “A Descriptor System Approach to Nonlinear Singularly Perturbed Optimal Control Problem.” Automatica, vol. 37, pp. 543-549, 2001.

Appendix A: The relation between the P(x) and $P_{new}(x_{new})$

In order to compute the optimal control law, the relations between the Riccati matrices $P(x)=\begin{bmatrix} P_{11}(x_1,x_2) & \varepsilon P_{21}{}^T(x_1,x_2) \\ P_{21}(x_1,x_2) & P_{22}(x_1,x_2)\end{bmatrix}$ and $P_{new}(x_{new})=\begin{bmatrix} P_s(x_s,x_f) & \varepsilon P_a(x_s,x_f) \\ \varepsilon P_b(x_s,x_f) & P_f(x_s,x_f)\end{bmatrix}$ must be determined. Suppose that $H_{22}{}^{-1}H_{21}=\begin{bmatrix} (l_{11})_{n_2\times n_1} & (l_{12})_{n_2\times n_1} \\ (l_{21})_{n_2\times n_1} & (l_{22})_{n_2\times n_1}\end{bmatrix}$, according to (17), we have:

$$\begin{cases} x_f = (l_{11}+l_{12}P_{11})x_1 + \left(I+\varepsilon l_{12}P_{21}{}^T\right)x_2, \\ p_s = P_{11} - \varepsilon\left(I+\varepsilon l_{12}P_{21}{}^T\right)^{-1} P_{21}{}^T (l_{11}+l_{12}P_{11}) + \left[O(\varepsilon^2)\right]_{n_1\times n_1}, \\ \varepsilon p_a = \varepsilon (P_{22}l_{12}-l_{22})^{-1} P_{21}{}^T (l_{11}+l_{12}P_{11}) + \left[O(\varepsilon^2)\right]_{n_1\times n_2}, \\ \varepsilon p_b = \varepsilon\left(I+\varepsilon l_{12}P_{21}{}^T\right)^{-1} P_{21}{}^T, \\ p_f = \left(P_{22}+\varepsilon l_{22}P_{21}{}^T\right)\left(I+\varepsilon l_{12}P_{21}{}^T\right)^{-1}, \end{cases} \tag{A1}$$

Then, for ε=0, one can write:

$$\begin{bmatrix} I_{n_1\times n_1} \\ P_s(x_s,x_f)\end{bmatrix} x_s = \begin{bmatrix} I_{n_1\times n_1} \\ P_{11}(x_1,x_2)\end{bmatrix} x_1 \tag{A2a}$$

$$\begin{bmatrix} I_{n_2\times n_2} \\ P_f(x_s,x_f) \end{bmatrix} x_f = H_{22}^{-1}H_{21}\begin{bmatrix} I_{n_1\times n_1} \\ P_{11}(x_1,x_2) \end{bmatrix} x_1 + \begin{bmatrix} 0_{n_2\times n_1} \\ P_{21}(x_1,x_2) \end{bmatrix} x_1 + \begin{bmatrix} I_{n_2\times n_2} \\ P_{22}(x_1,x_2) \end{bmatrix} x_2 \tag{A2b}$$

Now, multiplying (A2b) by $\begin{bmatrix} -P_f(x_s,x_f) & I_{n_2\times n_2} \end{bmatrix}$, the following relation is obtained.

$$\left(\begin{bmatrix} -P_f(x_s,x_f) & I_{n_2\times n_2} \end{bmatrix} H_{22}^{-1}H_{21}\begin{bmatrix} I_{n_1\times n_1} \\ P_{11}(x_1,x_2) \end{bmatrix} + P_{21}(x_1,x_2)\right)x_1 + \left(-P_f(x_s,x_f) + P_{22}(x_1,x_2)\right)x_2 = 0_{n_2} \tag{A3}$$

In other words, we have:

$$x_1 = x_s + [O(\varepsilon)]_{n_1} \tag{A4a}$$

$$P_{11}(x_1,x_2) = P_s(x_s,x_f) + [O(\varepsilon)]_{n_1\times n_1} \tag{A4b}$$

$$P_{22}(x_1,x_2) = P_f(x_s,x_f) + [O(\varepsilon)]_{n_2\times n_2} \tag{A4c}$$

$$P_{21}(x_1,x_2) = P_c(x_s,x_f) + [O(\varepsilon)]_{n_2\times n_1} \tag{A4d}$$

Where, $P_c(x_s,x_f) = \begin{bmatrix} P_f(x_s,x_f) & -I_{n_2\times n_2} \end{bmatrix} H_{22}^{-1}H_{21}\begin{bmatrix} I_{n_1\times n_1} \\ P_{11}(x_1,x_2) \end{bmatrix}$. Also, for ε=0, we have:

$$x_{1o} = x_{so} \tag{A5a}$$

$$P_{11o}(x_{1o},x_{2o}) = P_{so}(x_{so},x_{fo}) \tag{A5b}$$

$$P_{22o}(x_{1o},x_{2o}) = P_{fo}(x_{so},x_{fo}) \tag{A5c}$$

$$P_{21o}(x_{1o},x_{2o}) = P_{co}(x_{so},x_{fo}) \tag{A5d}$$

Appendix B: Proof of Theorem 1

a) The optimal control law

According to *λ=P(x)x* [3] and (A4), substituting Riccati matrices in (6c), the optimal control law would result as in (23).

b) The slow manifolds in boundary-layer correction

According to the singular perturbation theory, for *ε=0*, the fast variable should be derived with respect to the slow variable. Substituting *ε=0* in (19), the outer-layer equations are obtained as follows:

$$\dot{\chi}_{so} = H_{s\ |\varepsilon=0}\chi_{so} + H_{12o}\chi_{fo}, \tag{B1a}$$

$$0_{2n_2} = H_{22o}\chi_{fo}. \tag{B1b}$$

Substituting (17b) in (B1b), the following relation is derived:

$$H_{21o}X_{so} + H_{22o}X_{fo} = 0_{2n_2}. \tag{B2}$$

In other words, considering (14), we have:

$$\left(A_{21o} - S_{21o}P_{11o} - S_{22o}P_{21o}\right)x_{1o} + \left(A_{22o} - S_{22o}P_{22o}\right)x_{2o} = 0_{n_2}, \tag{B3a}$$

$$\left(A_{22o}{}^T - P_{22o}S_{22o}\right)P_{21o} + \left(A_{12o}{}^T - P_{22o}S_{21o}\right)P_{11o} + P_{22o}A_{21o} + Q_{21o} = 0_{n_1\times n_2}, \tag{B3b}$$

$$P_{22o}A_{22o} + A_{22o}{}^{T}P_{22o} - P_{22o}S_{22o}P_{22o} + Q_{22o} = 0_{n_2 \times n_2}, \tag{B3c}$$

For which, $x^*_{2o}(x_{1o},P_{11o})$, $P^*_{21o}(x_{1o},P_{11o})$ and $P^*_{22o}(x_{1o},P_{11o})$ are the solutions. The necessary conditions for (B3) to be solvable, $\{A_{22o}(x_{1o},x^*_{2o}),\ B_{2o}(x_{1o},x^*_{2o}),\ (Q_{22o}(x_{1o},x^*_{2o}))^{1/2}\}$ should be pointwise stabilizable and detectable for $\forall\left(x_{1o},x^*_{2o}\right) \in R^{n_1} \times R^{n_2}$ [2].

In (B1a), $H_{s\ |\varepsilon=0}$ for inside and out of the fast manifold, is separated as follows:

$$H_{s\ |\varepsilon=0} = H_{11}(x_1,x_2) - H_{12}(x_1,x_2)H_{22}{}^{-1}(x_1,x_2)H_{21}(x_1,x_2)_{|\varepsilon=0} =$$

$$\begin{bmatrix} A_s(x_{1o},x_2) & -S_s(x_{1o},x_2) \\ -Q_s(x_{1o},x_2) & -\left(A_s(x_{1o},x_2)\right)^T \end{bmatrix}, \quad t_0 \le t \le t_0 + \varepsilon t_1, \tag{B4a}$$

$$\begin{bmatrix} A_{so}(x_{1o},x^*_{2o}) & -S_{so}(x_{1o},x^*_{2o}) \\ -Q_{so}(x_{1o},x^*_{2o}) & -\left(A_{so}(x_{1o},x^*_{2o})\right)^T \end{bmatrix}, \quad t_0 + \varepsilon t_1 \le t \le t_F. \tag{B4b}$$

Substituting (B4) in (B1a), we have:

$$\dot{x}_{1o} = \left(A_s(x_{1o},x_2) - S_s(x_{1o},x_2)P_{so}(x_{1o},x_2)\right)x_{1o},\ x_{1o}\ |_{t_0} = x_1(t_0), \quad t_0 \le t \le t_0 + \varepsilon t_1, \tag{B5a}$$

$$\begin{bmatrix} \dot{x}_{1o} \\ P_{so}\dot{x}_{1o} + \dot{P}_{so}x_{1o} \end{bmatrix} = \begin{bmatrix} A_{so}(x_{1o},x^*_{2o})x_{1o} - S_{so}(x_{1o},x^*_{2o})P_{so}(x_{1o},x^*_{2o})x_{1o} \\ -Q_{so}(x_{1o},x^*_{2o})x_{1o} - A_{so}{}^{T}(x_{1o},x^*_{2o})P_{so}(x_{1o},x^*_{2o})x_{1o} \end{bmatrix}, P_{so}\ |_{t_F} = P_{11}(t_F), \quad t_0 + \varepsilon t_1 \le t \le t_F. \tag{B5b}$$

Thus, assuming that $\{A_{so}(x_{1o},x^*_{2o}),\ B_{so}(x_{1o},x^*_{2o}),\ (Q_{so}(x_{1o},x^*_{2o}))^{1/2}\}$ is pointwise stabilizable-detectable for $\forall\left(x_{1o},x^*_{2o}\right) \in R^{n_1} \times R^{n_2}$ [2], with rearrangement of (B5b), the SDRE of the slow variable is obtained as (22a).

Remark 3: Note that under assumption of above, P_{so} is unique, symmetric, positive definite solution of the SDRE (22a) that produces a locally asymptotically stable closed loop solution [2]. Thus the closed-loop matrix $A_s(x_{1o},x_2)-S_s(x_{1o},x_2)P_{so}$ is pointwise Hurwitz for $\forall (x_{1o},x_2) \in \Omega_1 \times \Omega_2$. Here, $\Omega_1 \times \Omega_2$ is any region such that the Lyapunov function is locally Lipschitz around the origin.

c) The fast manifold in initial layer correction

Since the time scale will be changed as $\tau = \dfrac{t-t_0}{\varepsilon}$ in the initial layer correction, the time derivative in this scale will be changed as $\dfrac{(.)}{d\tau} = \varepsilon \dfrac{d(.)}{dt}$ in forward time. Considering (4b), we have:

$$\frac{dx_2}{d\tau} = A_{22}(x_{1o},x_2)x_2 + A_{21}(x_{1o},x_2)x_{1o} + B_2(x_{1o},x_2)u, \ \ x_2\ |_{t_0} = x_2(t_0) \tag{B6}$$

Substituting (23) in (B6), according to (A4) and (14), the fast state equation in initial layer is obtained as (21b).

d) The fast manifold in final layer correction

Since the time scale will be changed as $\sigma = \dfrac{t_F - t}{\varepsilon}$ in the final layer correction, the time derivative in this scale will be changed as $\dfrac{d(.)}{d\sigma} = -\varepsilon \dfrac{d(.)}{dt}$ in backward time:

$$\begin{cases} \dfrac{d\chi_s}{d\sigma} = -\varepsilon\left(H_{11} - H_{12}H_{22}{}^{-1}H_{21}\right)\chi_s - \varepsilon H_{12}\chi_f \\ \dfrac{d\chi_f}{d\sigma} = -\varepsilon H_{22}{}^{-1}H_{21}\left(H_{11} - H_{12}H_{22}{}^{-1}H_{21}\right)\chi_s - \varepsilon\left(-H_{22}{}^{-1}\dot{H}_{22}H_{22}{}^{-1}H_{21} + H_{22}{}^{-1}\dot{H}_{21}\right)\chi_s - \left(H_{22} + \varepsilon H_{22}{}^{-1}H_{21}H_{12}\right)\chi_f \end{cases} \tag{B7}$$

Substituting ε=0 in (B7), we have $\chi_s(\sigma) = 0_{2n_1}$. Therefore, the final layer correction equation is obtained as:

$$\frac{d\chi_f}{d\sigma} = -H_{22}(x_{1o}, x^*{}_{2o})\chi_f, \quad \chi_f|_{\sigma=0} = \chi_f(x_1(t_F), x_2(t_F)). \tag{B8}$$

Now, substituting (20b) and (17b) in (B8), we have:

$$\begin{bmatrix} \dfrac{dx_f}{d\sigma} \\ P_f \dfrac{dx_f}{d\sigma} + \dfrac{dP_f}{d\sigma} x_f \end{bmatrix} = -\begin{bmatrix} A_{22}(x_{1o}, x^*{}_{2o})x_f - S_{22}(x_{1o}, x^*{}_{2o})P_f x_f \\ -Q_{22}(x_{1o}, x^*{}_{2o})x_f - A_{22}{}^T(x_{1o}, x^*{}_{2o})P_f x_f \end{bmatrix}, P_f|_{t_F} = P_{22}(t_F), \quad t_F - \varepsilon t_2 \le t \le t_F. \tag{B9}$$

Thus, assuming that $\{A_{22o}(x_{1o},x^*{}_{2o}),\ B_{2o}(x_{1o},x^*{}_{2o}),\ (Q_{22o}(x_{1o},x^*{}_{2o}))^{1/2}\}$ is stabilizable-detectable for $\forall(x_{1o}, x^*{}_{2o}) \in R^{n_1} \times R^{n_2}$ [2], according to (A5) and (14), the SDRE of the fast variable is obtained as (22b).

Remark 4: Note that under assumption of above, P_f is unique, symmetric, positive definite solution of the SDRE (22b) that produces a locally asymptotically stable closed loop solution [2]. Thus, the closed-loop matrix $A_{22}(x_{1o},x_2)-S_{22}(x_{1o},x_2)P^*{}_{22o}$ is pointwise Hurwitz for $\forall(x_{1o},x_2) \in \Omega_1 \times \Omega_2$. Here, $\Omega_1 \times \Omega_2$ is any region such that the Lyapunov function is locally Lipschitz around the origin. □