Comparison of Three Weibull Extensions; Generalized Gamma, Exponentiated Weibull, and Odd Weibull Distributions

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Abstract

In this paper, three-parameter Weibull extensions: generalized gamma (GG), exponentiated Weibull (EW), and Odd Weibull (OW) distributions, which are capable of modeling data that exhibit five major hazard shapes, are compared using Vuong, empirical distribution function, and Shapiro-Wilk W- tests. The goal of this work is to compare the GG, EW, and OW distribution using better model selection criteria under general conditions as addressed by the Voung test, in addition to the tests considered. Our simulation study and graphical analysis show that the OW distribution is different from both GG and EW distributions even though all three distributions have five common hazard shapes. An example using voltage data is also considered to illustrate applications of the three distributions. Our comparative results led us to develop a six-parameter generalized distribution which is an extension of the GG, EW, and OW distributions. A four-parameter sub-model of this distribution, the exponentiated Odd Weibull distribution which has applications in modeling lifetime data, was studied in our recent publication.

Keywords: Exponentiated Odd Weibull Distribution, Simulation Study, Vuong Test, Empirical Distribution Function, Shapiro-Wilk W-test.

1. INTRODUCTION

In a recent article [1], we developed a six-parameter generalized Weibull distribution which is an extension of the generalized gamma (GG)[2], exponentiated Weibull (EW)[3], and Odd Weibull (OW)[4] distributions. We also studied one of its submodel, the exponentiated Odd Weibull distribution. In this study, we present results which motivated us to develop the six-parameter generalized Weibull distribution in [1]. Specifically, we compare the GG, EW, and OW distributions in order to investigate whether it is reasonable to develop an extension of the three distributions.

As mentioned in [1], the GG and EW distributions are well-known distributions in the literature, while the OW distribution has not yet been studied extensively. The OW distribution was developed by [4] considering the distributions of the odds of the Weibull and inverse Weibull distributions. The interest in considering the GG, EW, and OW distributions is that all three distributions are an extension of the Weibull distribution with an extra parameter. All three distributions have five common hazard shapes: constant, increasing, decreasing, bathtub, and arc-shape. In addition, the OW is very flexible in modeling lifetime data in that its hazard function exhibits 8 different shapes: constant, decreasing, increasing, arc-shape, bathtub, S-shape, inverse-S shape[5], and unimodal. Properties of the OW distribution are discussed in [4, 5, 6, 7].

In literature, the GG and EW distributions were compared by [8] based on both the Kullback-Leibler distances and quantile functions. The GG and EW were also compared by [9] using the quartile ratio relationship log(Q2/Q1)/log(Q3/Q2). [10] compared the hazard flexibility of the GG, EW, and OW distributions based on a criterion developed from the total time on test transform plot. Furthermore, the GG and OW distributions were compared by [7] using both likelihood ratio test and Cramér-von Mises-type distances. In this paper, we compare GG, EW, and OW distributions based on Vuong[11], empirical distribution function (EDF) tests [12], and Shapiro-Wilk W-test[13]. Specifically, we consider three EDF test statistics: Kolmogorov Smirnov (KS), Cram´er-von Mises (CM), and Anderson Darling (AD) tests. This criterion uses a simulation study to analyze the performance of different density and hazard shapes of the GG, EW, and OW distributions. The main purpose of this work is to compare the GG, EW, and OW distribution using better model selection criteria under general conditions, whether the competing models are nested, overlapping, or non-nested, and whether the models are correctly specified, as addressed by [11]. This criterion has the desirable property that it coincides with the usual classical testing approach when the models are nested [11]. In addition, we use the KS test since it measures good fit in the middle of the distribution. On the other hand, the AD test measures good fit in the tail area, while CM test measures the overall fit. Observe that the criteria used in this paper is different from the ones used by [7, 8, 10] since there is no need to repeat the same work.

In addition, the GG, OW, and EW distributions are compared graphically using the Shapiro-Wilk W-test statistic. An example on voltage data is used to illustrate applications of the GG, EW, and OW distributions.

2. BACKGROUND AND METHODS

The goal of this work is to first compare the performance of the GG, EW, and OW distributions using a deductive research method. This is achieved through a simulation study based on data generated from the three distributions. Secondly, the three distributions are compared graphically using the Shapiro-Wilk W-test. This is comparative research since this study is performed by comparing data from the three distributions considered. The cumulative distribution functions of the GG, EW, and OW distributions are, respectively,

$$F_{GG}(x; \lambda, \theta, k) = \Gamma\{ (x/\theta)^{\lambda} , k\} ; \lambda > 0, \theta > 0, k > 0,$$

$$F_{EW}(x; \lambda, \theta, \gamma) = [1 - \exp\{-(x/\theta)^{\lambda}\}]^{\gamma} ; \lambda > 0, \theta > 0, \gamma > 0,$$

$$F_{OW}(x; \lambda, \beta, \theta) = 1 - (1 + [\exp\{(x/ \theta)^{\lambda}\} - 1]^{\beta})^{-1} ; \lambda\beta>0, \theta>0,$$

where $x > 0$; $\Gamma(t; k)$ is the incomplete gamma function defined by $\Gamma(t,k) = (1/\Gamma(k)) \int_0^t x^{k-1} exp(-x)dx, k > 0$, and $\Gamma(k)$ is the complete gamma function defined by $\Gamma(k) = \int_0^{\infty} t^{k-1} exp(-t)dt$.

2.1 Comparison using Vuong and Empirical Distribution Tests

2.1.1 General Approach

Pairwise comparisons of the GG, EW, and OW distributions are performed by using Vuong[11], KS, CM, and AD tests[12]. We compare the three distributions when the data are generated from the GG, EW, and OW distributions, respectively. We simulate data from the GG, EW, and OW distributions with sample sizes n = {100, 200, 400, 800} to produce different density and hazard shapes such that f (x) = {unimodal (U), bimodal (BI)}, and h(x) = {decreasing (D), increasing (I), arc-shape (A), bathtub (B), S-shape (S), inverse-S shape (IS) , unimodal low-peak (U_{lp}), unimodal high-peak (U_{hp}) . For each pairwise comparison, the test statistics for Vuong, KS, CM, and AD tests are computed at a significance level of $\alpha = 0.05$, based on 1000 replications.

2.1.2 Hypothesis Test

Denote the distributions GG, EW, and OW by F_j, $j = 1, 2, 3$, respectively; and their corresponding pdfs by f_j, $j = 1, 2, 3$, respectively. Given the two distributions F_j and F_k for a fixed $j \neq k$ with $j, k = 1, 2, 3$, we consider the hypotheses, H_0 : Distributions F_j and F_k are equivalent against H_{fj} : Distribution F_j is better than distribution F_k, or H_{fk} : Distribution F_k is better than distribution F_j. Observe that distributions are equivalent means the distributions are equally closer to the data generating distribution. Note that when the data are coming from the distribution F_j, we record the proportion of times when both distributions F_j and F_k are equivalent (P_e), and the proportion of times when F_j is better than F_k. The proportion of times when F_k is better than F_j can be calculated as $1 - P_e - P_{Fj}$. In Table 1, P_G is the proportion of times when GG is better, P_E is the proportion of times when EW is better, and P_O is the proportion of times when OW is better. An example of the omitted proportion is when the data are coming from GG compared with EW; the proportion of times when EW is better than GG is $1 - P_e - P_G$. Results are summarized in Table 1. Observe that we did not consider any variation of the scale parameter θ when we simulate data from all three distributions. Since it does not affect the shape of the density or hazard functions. Specifically, in this simulation study, 9 parameters of the 3 distributions are simultaneously estimated to compute the test statistics defined in this section.

2.1.3 Description of Vuong Test

Pairwise comparisons of the GG, EW, and OW distributions are performed by using a test statistic proposed by [11] for non-nested distributions. The likelihood ratio statistic for testing distribution F_j against distribution F_k is

$$L_* = \sum_{i=1}^{n} \log\left(\frac{f_j(x_i;\, \boldsymbol{\theta}_j)}{f_k(x_i;\, \boldsymbol{\theta}_k)}\right).$$

Since the GG, EW, and OW distributions are non-nested, the statistic L_* is not chi-squared distributed.

Vuong used the Kullback-Liebler information criteria to develop the following test statistic T_* to discriminate between two non-nested distributions. To test the null hypothesis H_0, [11] proposed the statistic $T_* = L_*/\hat{\omega}\sqrt{n}$ where ω^2 is the estimate of the variance of $L_*/\sqrt{n}$ such that:

$$\omega^2 = (1/n)\sum_{i=1}^{n}\{\log\big(f_j(x_i;\, \boldsymbol{\theta}_j)/f_k(x_i;\, \boldsymbol{\theta}_k)\big)\}^2 - (L_*/n)^2.$$

We use Vuong test because it is less computationally intensive for parametric distributions than computing Kullback-Liebler distances between densities. For non-nested distributions, T_* is approximately standard normally distributed under the null hypothesis that the two distributions are equivalent. At significance level α, we compare T_* with $z_{\alpha/2}$. If $T_* < -z_{\alpha/2}$, the null hypothesis is rejected in favor of H_{f_k} , the distribution that F_k is better than F_j. If $T_* > z_{\alpha/2}$, the null hypothesis is rejected in favor of H_{f_j} , the distribution that F_j is better than F_k. However, if $|T_*| \leq z_{\alpha/2}$, the null hypothesis is not rejected, hence, there is no evidence to say that both distributions are not equivalent.

2.1.4 Description of EDF Tests

Pairwise comparisons of the GG, EW, and OW distributions are also performed using the EDF test statistics: KS, CM, and AD tests[12]. As outlined in [12], the test statistics for each test are defined as follows. The formulas involve the cdf values arranged in ascending order, $F(x_1) < F(x_2) \cdot \cdot \cdot < F(x_n)$.

The KS test statistic is $D = \max(D^+, D^-)$, where $D^+ = \max_{1\le i\le n}(i/n - F(x_i))$, and $D^- = \max_{1\le i\le n}\{F(x_i) - (i-1)/n\}$. The CM test statistic is

$W^2 = \sum_i \{F(x_i) - (2i-1)/(2n)\}^2 + 1/(12n)$. The AD test statistic is $A^2 = -n - (1/n)\sum_i \{(2i-1)\ log\ F(x_i) + (2n+1-2i)\ log\ (1 - F(x_i))\}$.

We use the KS test statistics because it places more emphasis on good fit in the middle of the distribution. On the other hand, the AD test statistic places more emphasis on good fit in the tails than in the middle of the distribution, while the CM test focuses on the overall fit of the distribution.

2.2 Graphical Comparison using The Shapiro-Wilk W-test

The performance of the GG, EW, and OW distributions was compared graphically using the Shapiro- Wilk W-test [13] for assessing normality of the distributions. The Shapiro-Wilk W-test statistic is given by

$$W = \sum_{i=1}^{n} (a_{i,n} x_{(i)})^2 / \sum_{i=1}^{n} (x_i - \bar{x})^2,$$

where $x_{(1)}, x_{(2)}, \ldots, x_{(3)}$ are sample order statistics, and the weights $a_{i,n}$ were obtained from [14].

For each distribution, sample data were generated from the uniform distribution, U(0,1) using sample sizes n = 25, 30, 40, and 50, and then plugged into its quantile function. Values of the W-test statistic were computed by considering different values of shape parameters for the distribution. Results for the surfaces and corresponding contour plots are shown in Figures 1 − 3.

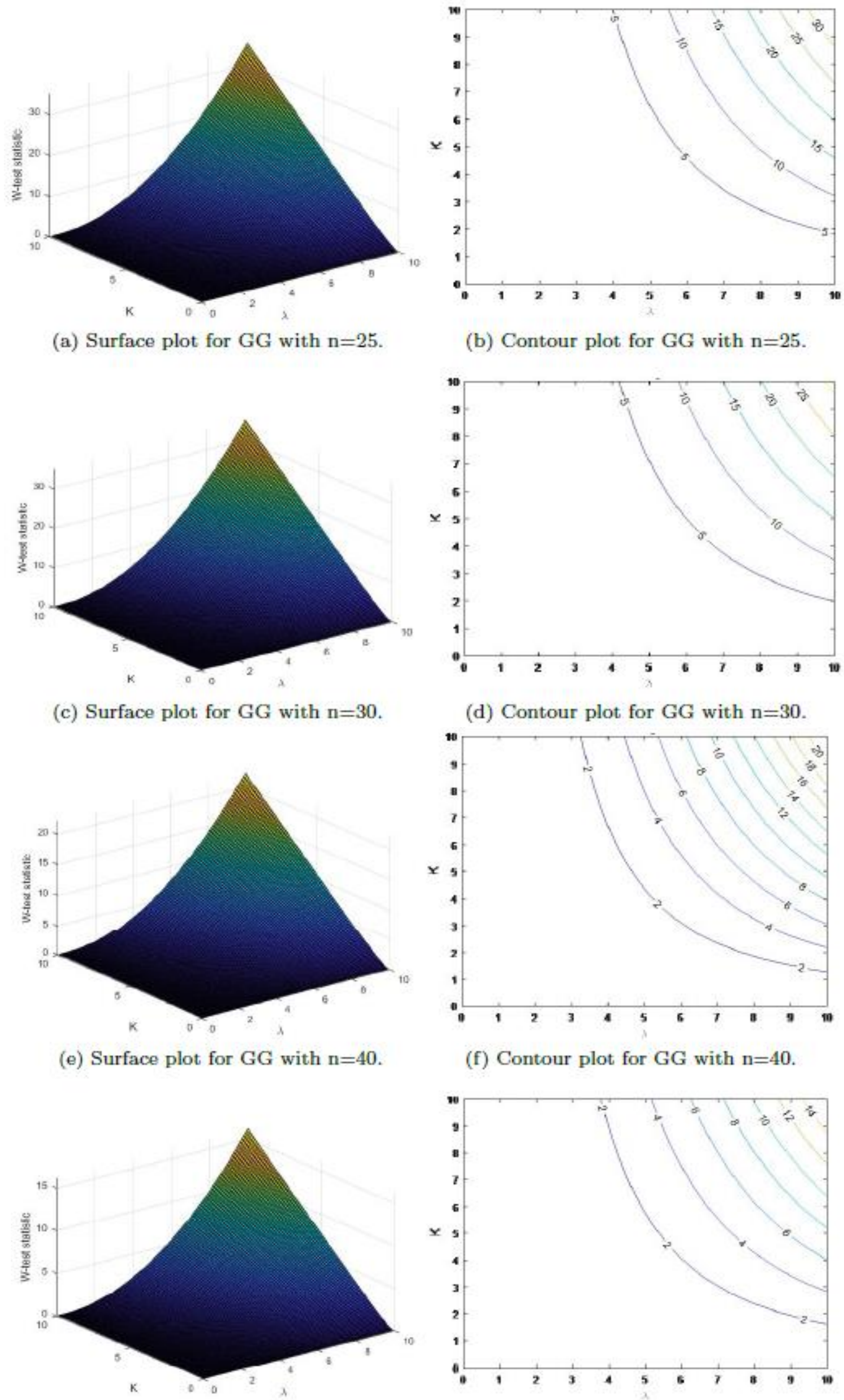
3. RESULTS AND DISCUSSION

In Table 1, when the data are coming from the GG compared with the EW distribution, both Vuong and KS tests show that GG and EW are more equivalent as compared to the results from AD test. This result further justifies that the GG and EW are more similar in the middle of the distributions than in the tail area. Similar results are observed when the data are coming from the EW distribution. Based on our simulation results, when the data are coming from the OW and the hazard function is S-shape, the OW is not significantly different from either GG or EW distribution as the sample size n becomes larger. The reason is that the S- shape of the OW is not very prominent. However, when the data are coming from the OW distribution and the hazard function is increasing, bathtub, inverse-S shape, unimodal low-peak, or unimodal high-peak, the OW is significantly different from the GG and EW as n becomes larger based on all the four tests. Clearly, in this case the OW density function is either bimodal or unimodal with a thick upper tail.

Parameters ($\theta = 1$)	$h(x)$	$f(x)$	n	Vuong test				KS test				CM test				AD test			
Data coming from GG				C. OW		C. EW		C. OW		C. EW		C. OW		C. EW		C. OW		C. EW	
				P_G	P_e	P_G	P_e	P_G	P_e	P_G	P_e	P_G	P_e	P_G	P_e	P_G	P_e	P_G	P_e
$\lambda = 0.5, k = 0.5$	D	U	100	.00	.99	.00	.99	.04	.89	.05	.89	.07	.87	.05	.90	.07	.87	.04	.90
			200	.01	.97	.00	.99	.07	.87	.05	.89	.11	.84	.04	.90	.12	.82	.04	.90
			400	.02	.97	.01	.98	.08	.86	.05	.89	.13	.82	.04	.90	.14	.82	.03	.91
			800	.06	.93	.02	.97	.12	.85	.04	.90	.24	.74	.04	.92	.25	.73	.03	.92
$\lambda = 5, k = 2$	I	U	100	.06	.91	.00	.98	.03	.89	.02	.92	.00	.90	.00	.94	.02	.91	.00	.94
			200	.06	.94	.01	.99	.02	.91	.03	.90	.02	.92	.02	.93	.02	.93	.01	.93
			400	.06	.94	.01	.99	.06	.90	.04	.91	.05	.90	.03	.90	.06	.89	.03	.90
			800	.11	.89	.02	.98	.09	.88	.04	.90	.07	.88	.04	.90	.10	.86	.03	.91
$\lambda = 0.6, k = 5$	A	U	100	.09	.91	.02	.99	.03	.91	.04	.90	.03	.91	.03	.90	.04	.89	.03	.89
			200	.12	.88	.01	.99	.04	.89	.03	.89	.04	.90	.02	.89	.05	.90	.02	.89
			400	.20	.81	.02	.98	.09	.88	.03	.90	.09	.89	.02	.90	12	.86	.02	.89
			800	.32	.68	.02	.98	.29	.70	.04	.89	.21	.78	.02	.92	.29	.71	.02	.91
$\lambda = 1.2, k = 0.6$	B	U	100	.05	.93	.00	.97	.09	.85	.03	.92	.09	.85	.03	.92	.09	.86	.03	.92
			200	.01	.99	.00	.99	.06	.90	.05	.91	.09	.88	.05	.91	.08	.88	.05	.90
			400	.01	.98	.00	.99	.07	.89	.05	.90	.10	.88	.05	.90	.10	.87	.05	.90
			800	.02	.97	.00	.97	.10	.86	.04	.90	.13	.84	.04	.90	.14	.83	.04	.91
Data coming from EW				C. OW		C. GG		C. OW		C. GG		C. OW		C. GG		C. OW		C. GG	
				P_E	P_e	P_E	P_e	P_E	P_e	P_E	P_e	P_E	P_e	P_E	P_e	P_E	P_e	P_E	P_e
$\lambda = 0.5, \gamma = 0.5$	D	U	100	.01	.99	.04	.96	.05	.90	.06	.89	.06	.89	.06	.90	.07	.88	.07	.89
			200	.03	.97	.02	.97	.07	.89	.04	.90	.11	.84	.05	.91	.12	.84	.05	.90
			400	.03	.97	.02	.97	.08	.88	.05	.90	.16	.81	.05	.90	.15	.81	.06	.90
			800	.08	.92	.03	.96	.16	.81	.05	.91	.28	.69	.06	.89	.29	.69	.07	.89
$\lambda = 5, \gamma = 2$	I	U	100	.02	.98	.01	.98	.05	.90	.09	.87	.05	.90	.11	.86	.05	.91	.11	.86
			200	.04	.96	.01	.99	.06	.89	.09	.88	.06	.89	.08	.88	.07	.89	.09	.87
			400	.07	.93	.00	1.0	.08	.88	.08	.88	.09	.87	.10	.87	.11	.85	.10	.87
			800	.13	.87	.00	.99	.13	.86	.10	.87	.13	.84	.13	.84	.18	.80	.15	.83
$\lambda = 0.5, \gamma = 5$	A	U	100	.08	.92	.00	.99	.05	.91	.07	.90	.05	.91	.07	.91	.07	.89	.08	.89
			200	.20	.80	.00	1.0	.07	.90	.07	.90	.09	.86	.09	.88	.13	.84	.11	.87
			400	.32	.68	.00	1.0	.15	.84	.14	.84	.15	.84	.21	.77	.23	.77	.25	.74
			800	.49	.51	.01	.99	.43	.57	.29	.70	.39	.60	.42	.58	.55	.45	.47	.53
$\lambda = 3, \gamma = 0.2$	B	U	100	.04	.95	.04	.95	.03	.91	.08	.87	.09	.84	.07	.87	.07	.87	.07	.88
			200	.05	.93	.02	.96	.03	.92	.07	.89	.10	.85	.04	.91	.10	.87	.05	.91
			400	.12	.87	.01	.94	.02	.94	.03	.93	.11	.88	.03	.93	.10	.89	.03	.94
			800	.21	.78	.05	.87	.00	.98	.02	.95	.07	.93	.02	.95	.06	.94	.03	.95

Parameters ($\theta = 1$)	$h(x)$	$f(x)$	n	Vuong test				KS test				CM test				AD test			
Data coming from OW				C. EW		C. GG		C. EW		C. GG		C. EW		C. GG		C. EW		C. GG	
				P_O	P_e	P_O	P_e	P_O	P_e	P_O	P_e	P_O	P_e	P_O	P_e	P_O	P_e	P_O	P_e
$\lambda = 0.5, \beta = 0.5$	D	U	100	.05	.95	.18	.82	.06	.92	.27	.72	.05	.91	.24	.72	.06	.91	.30	.66
			200	.10	.89	.23	.77	.13	.85	.33	.65	.08	.90	.25	.73	.10	.87	.28	.70
			400	.14	.86	.24	.76	.14	.86	.46	.53	.12	.87	.34	.65	.18	.81	.41	.58
			800	.25	.76	.48	.53	.30	.70	.92	.08	.35	.65	.80	.20	.42	.58	.84	.16
$\lambda = 10, \beta = 0.5$	I	U	100	.05	.95	.81	.19	.08	.90	.98	.03	.05	.92	.92	.08	.07	.90	.97	.03
			200	.09	.91	.99	.01	.11	.87	1.0	.00	.06	.92	1.0	.00	.09	.89	1.0	.00
			400	.16	.84	1.0	.00	.16	.83	1.0	.00	.16	.84	1.0	.00	.20	.79	1.0	.00
			800	.25	.75	1.0	.00	.30	.70	1.0	.00	.30	.70	1.0	.00	.42	.58	1.0	.00
$\lambda = 0.5, \beta = 2$	A	U	100	.01	.97	.01	.96	.17	.80	.20	.76	.15	.82	.21	.77	.15	.82	.23	.76
			200	.03	.96	.04	.95	.21	.77	.28	.70	.20	.76	.30	.68	.20	.77	.32	.66
			400	.06	.92	.09	.91	.30	.69	.38	.61	.29	.69	.40	.60	.30	.69	.40	.60
			800	.15	.85	.19	.81	.53	.47	.64	.36	.49	.51	.59	.41	.53	.47	.65	.35
$\lambda = 0.6, \beta = 1.6$	S	U	100	.04	.96	.01	.98	.15	.81	.20	.78	.13	.84	.22	.75	.13	.82	.24	.73
			200	.01	.99	.01	.99	.13	.84	.27	.72	.12	.84	.28	.72	.12	.84	.30	.69
			400	.02	.98	.04	.96	.17	.80	.67	.33	.72	.80	.72	.28	.17	.81	.77	.23
			800	.05	.95	.72	.28	.26	.72	1.0	.00	.28	.71	1.0	.00	.28	.70	1.0	.01
$\lambda = -0.5, \beta = -2$	U_{lp}	U	100	.33	.66	61	.39	.77	.22	.93	.07	.77	.23	.95	.05	.75	.25	.96	.04
			200	.43	.57	.84	.16	.86	.14	1.0	.00	.88	.12	1.0	.00	.88	.12	1.0	.00
			400	.84	.16	.99	.01	1.0	.00	1.0	.00	1.0	.00	1.0	.00	1.0	.00	1.0	.00
			800	.99	.00	1.0	.00	1.0	.00	1.0	.00	1.0	.00	1.0	00	1.0	.00	1.0	.00
$\lambda = -2, \beta = -0.5$	U_{hp}	U	100	.98	.02	.99	.01	1.0	.01	1.0	.00	1.0	.00	1.0	.00	1.0	.00	1.0	.00
			200	1.0	.01	1.0	.00	1.0	.00	1.0	.00	1.0	.00	1.0	.00	1.0	.00	1.0	.00
			400	1.0	.01	1.0	.00	1.0	.00	1.0	.00	1.0	.00	1.0	.00	1.0	.00	1.0	.00
			800	1.0	.01	1.0	.00	1.0	.00	1.0	.00	1.0	.00	1.0	.00	1.0	.00	1.0	.00
$\lambda = 8, \beta = 0.2$	I	BI	100	.25	.75	.92	.08	.58	.42	.93	.07	.45	.55	.91	.09	.53	.47	.92	.08
			200	.86	.15	1.0	.00	.99	.01	1.0	.00	.96	.04	1.0	.00	.98	.02	1.0	.00
			400	1.0	.00	1.0	.00	1.0	.00	1.0	.00	1.0	.00	1.0	.00	1.0	.00	1.0	.00
			800	1.0	.00	1.0	.00	1.0	.00	1.0	.00	1.0	.00	1.0	.00	1.0	.00	1.0	.00
$\lambda = 4, \beta = 0.2$	B	BI	100	.30	.70	.99	.01	.68	.32	.99	.01	.52	.48	.98	.02	.63	.37	.99	.01
			200	.75	.25	1.0	.00	.97	.03	1.0	.00	.90	.10	1.0	.00	.94	.06	1.0	.00
			400	1.0	.01	1.0	.00	1.0	.00	1.0	.00	1.0	.00	1.0	.00	1.0	.00	1.0	.00
			800	1.0	.01	1.0	.00	1.0	.00	1.0	.00	1.0	.00	1.0	.00	1.0	.00	1.0	.00
$\lambda = 3.5, \beta = 0.3$	IS	BI	100	.06	.94	.94	.06	.16	.84	.95	.05	.06	.92	.93	.07	.10	.90	.95	.05
			200	.22	.78	.99	.01	.39	.61	.99	.01	.21	.79	.99	.01	.33	.67	.99	.01
			400	.40	.60	1.0	.00	.69	.31	1.0	.00	.49	.51	1.0	.00	.59	.41	1.0	.00
			800	.88	.12	1.0	.00	.99	.01	1.0	.00	.95	.05	1.0	.00	97	.03	1.0	.00

TABLE 1: Vuong, KS, CM, and AD tests for GG, EW, and OW distributions based on 1000 simulations.



(a) Surface plot for GG with n=25.

(b) Contour plot for GG with n=25.

(c) Surface plot for GG with n=30.

(d) Contour plot for GG with n=30.

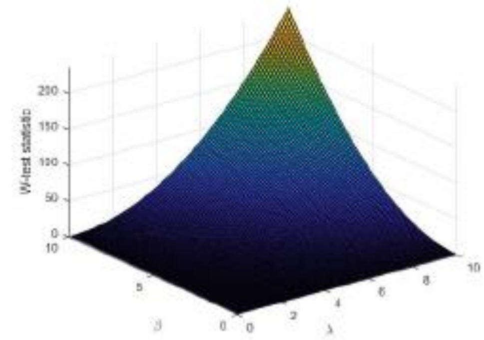
(e) Surface plot for GG with n=40.

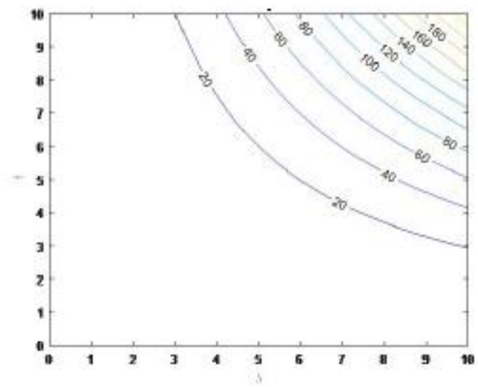
(f) Contour plot for GG with n=40.

(g) Surface plot for GG with n=50.

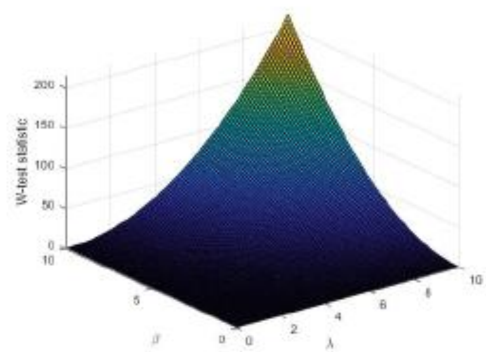
(h) Contour plot for GG with n=50.

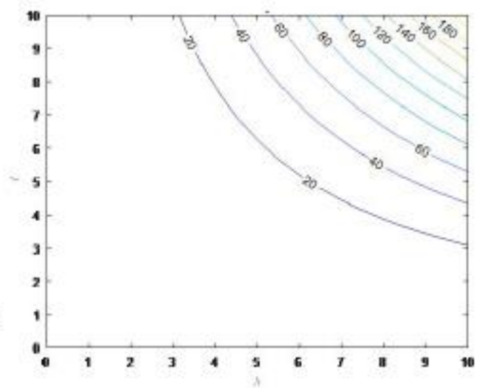
FIGURE 1: W-test statistic plots for GG using sample sizes 25, 30, 40, and 50.



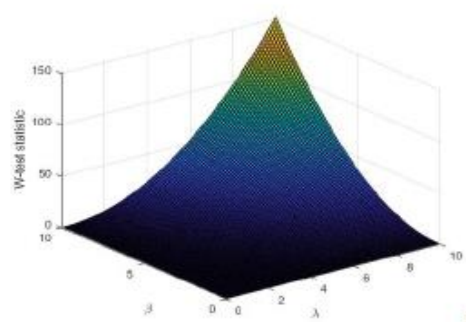


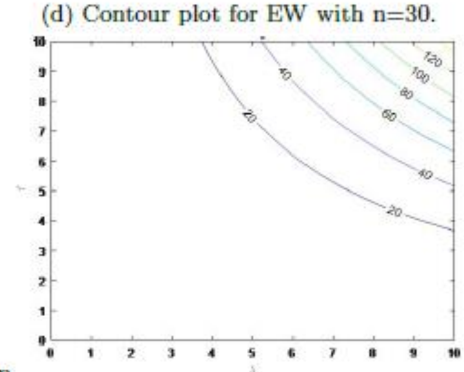
(a) Surface plot for EW with n=25. (b) Contour plot for EW with n=25.





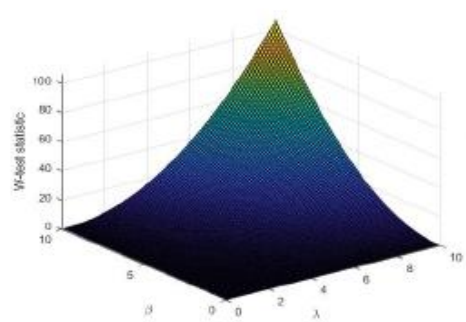
(c) Surface plot for EW with n=30. (d) Contour plot for EW with n=30.

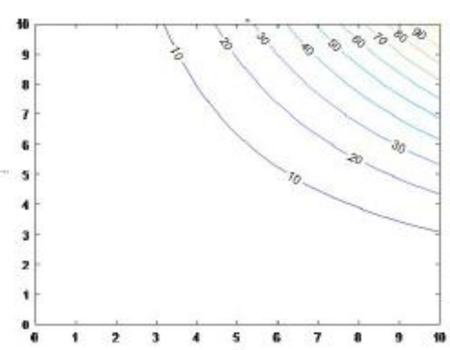




nn

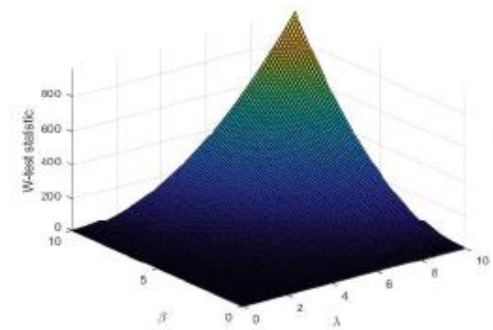
(e) Surface plot for EW with n=40. (f) Contour plot for EW with n=40.

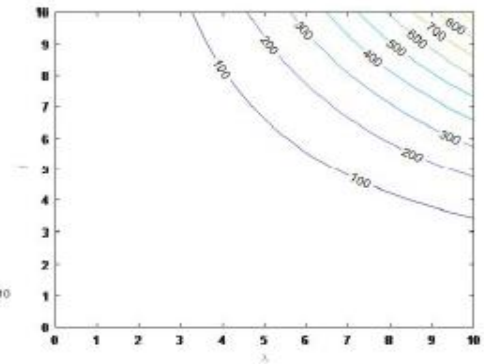




(g) Surface plot for EW with n=50. (h) Contour plot for EW with n=50.

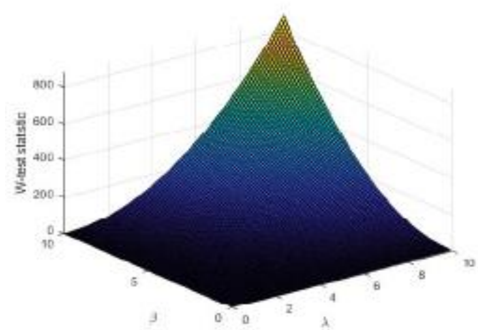
FIGURE 2: W-test statistic plots for EW using sample sizes 25, 30, 40, and 50.

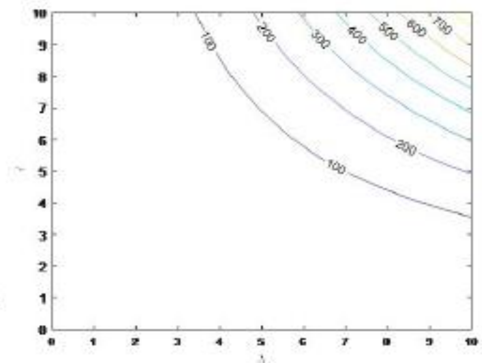




(a) Surface plot for OW with n=25.

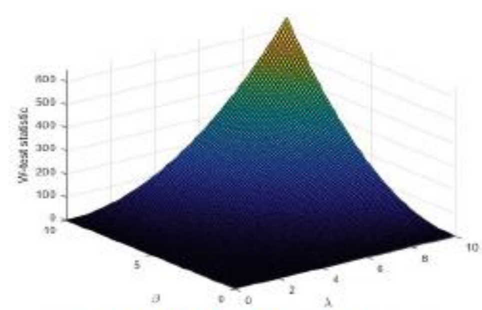
(b) Contour plot for OW with n=25.

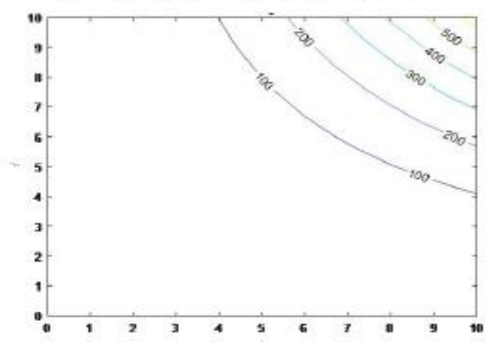




(c) Surface plot for OW with n=30.

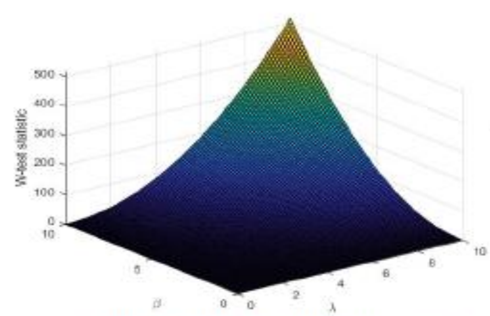
(d) Contour plot for OW with n=30.

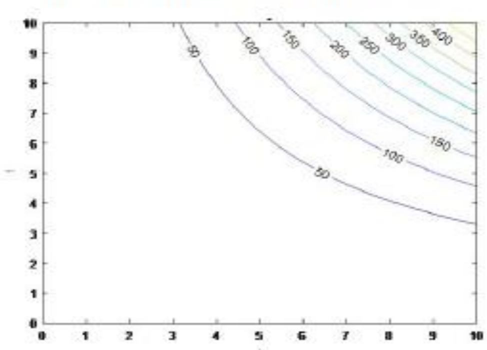




(e) Surface plot for OW with n=40.

(f) Contour plot for OW with n=40.





(g) Surface plot for OW with n=50.

(h) Contour plot for OW with n=50.

FIGURE 3: W-test statistic plots for OW using sample sizes 25, 30, 40, and 50.

The proportion of times the distributions are equivalent goes to zero as n becomes larger whenever OW is compared with either GG or EW distributions. Clearly, these proportions go to zero faster when OW is compared with GG than the EW distribution. Examples are the bathtub and inverse-S hazard shapes, which highlight the difference in performance between the GG and EW distributions. These results show that the GG and EW distributions provide a poor fit for the simulated data if data are coming from the OW distribution. In the cases considered, the GG performs poor than the EW distribution even though the two distributions have the same hazard taxonomy. Differences in performance of the GG and EW are also observed on the arc-shape hazard function when the data are coming from EW and compared with the GG distribution based on KS, CM, and AD tests.

These results show differences between the performance of the GG and EW distributions since our analysis is based on better model selection criteria under general conditions through the tests considered. The Voung test used in this analysis is an improvement of the Kullback Leibler distances used by [8] since it is a better model selection criteria under general conditions and is based on both Kullback Leibler distances and likelihood ratio test statistic. By general conditions, this refers to whether the competing models are nested, overlapping, or non-nested, and whether the models are correctly specified. Vuong test is less computationally intensive for parametric distributions than computing Kullback-Liebler distances between densities. Moreover, it also served as a better selection criteria since it is probabilistic and is based on testing if the competing models are as close to the true distribution against the hypothesis that one model is closer than the other [11]. The KS, AD, and CM tests provide additional support for the differences in the performance of the GG, EW, and OW distributions by highlighting differences in fit in the middle of the distribution, the tail area, and the overall fit, respectively, between the pairwise distribution comparisons considered. Thus, our analysis is an improvement to work previously done in literature.

Furthermore, from Figures 1 - 3, we observe different patterns of departure from normality for the three distributions based on the Shapiro-Wilk W-test statistic. As illustrated by the surface plots and contour plots, the maximum value of the Shapiro-Wilk W-test statistic achieved by each distribution is different from the maximum values achieved by the other two distributions. The GG achieves the smallest maximum value while the OW achieves the largest maximum value of the test statistic. This shows that the OW results in samples which are further away from normality than the GG and EW. As the sample size increases, the maximum value achieved by the test statistic decreases for each distribution. Although this value remains significantly different from the maximum values achieved by the other two distributions. The surface plots also illustrate the differences in the steepness of the graphs, with the GG resulting in a more steeper graph than the EW and OW distributions. Based on our analysis, we conclude that the OW distribution is significantly different from GG and EW distributions.

4. APPLICATIONS: VOLTAGE DATA EXAMPLE

For application purposes, the GG, EW, and OW distributions are fitted to the voltage data studied by [15], which gives failure times and running times for a sample of devices from a field-tracking study of a larger system. The dataset consists of 30 units which were initially installed in normal service conditions. The two causes of failure observed for each unit that failed were accumulation of randomly occurring damage from power-line voltage spikes during electric storms, and failure caused by normal product wear [15]. We analyze this dataset to investigate whether there is any significant difference in the fit of the models estimated by using the GG, EW, and OW distributions. Observe that we did not fit the models presented in [15] since we only considered three-parameter models in this study.

	GG	EW	OW
MLE (SE)	$\hat{\lambda}$ = 0.8037(0.2941) $\hat{k}$ = 1.6967(0.8168) $\hat{\theta}$= 81.217(72.0155)	$\hat{\lambda}$ = 7.1062(0.2387) $\hat{\gamma}$ = 0.1294(0.0246) $\hat{\theta}$= 322.02(23.167)	$\hat{\lambda}$ = 7.7817(0.2235) $\hat{\beta}$ = 0.1108(0.0242) $\hat{\theta}$= 205.68(7.5596)
NLL	188.91	177.05	**168.45**
AIC	377.82	360.10	**342.90**
BIC	382.02	364.30	**347.10**
KS	0.2210	0.2189	**0.1313**
CM	0.3423	0.2548	**0.0710**

TABLE 2: Estimates of fitted distributions for voltage data.

Parameters of the three distributions were estimated using the maximum likelihood estimation method. Our analysis is based on the Akaike information criteria (AIC), Bayesian information criteria (BIC), Kolmogorov Smirnov (KS) test statistic, and Cramer-von Mises (CM) type distances defined by the sum of the squared KS distances for each complete data. Results from the analysis are given in Table 2. The standard errors (SE) are given in parenthesis. From Table 2, the OW distribution gives the lowest AIC, BIC, KS, and CM values. Thus, it is the best distribution for fitting the voltage data, among the three distributions considered. It is followed by the EW distribution, which has the second smallest values for the AIC, BIC, KS, and CM test statistics. Observe that the parameter θ has a high standard error for the GG and EW distributions.

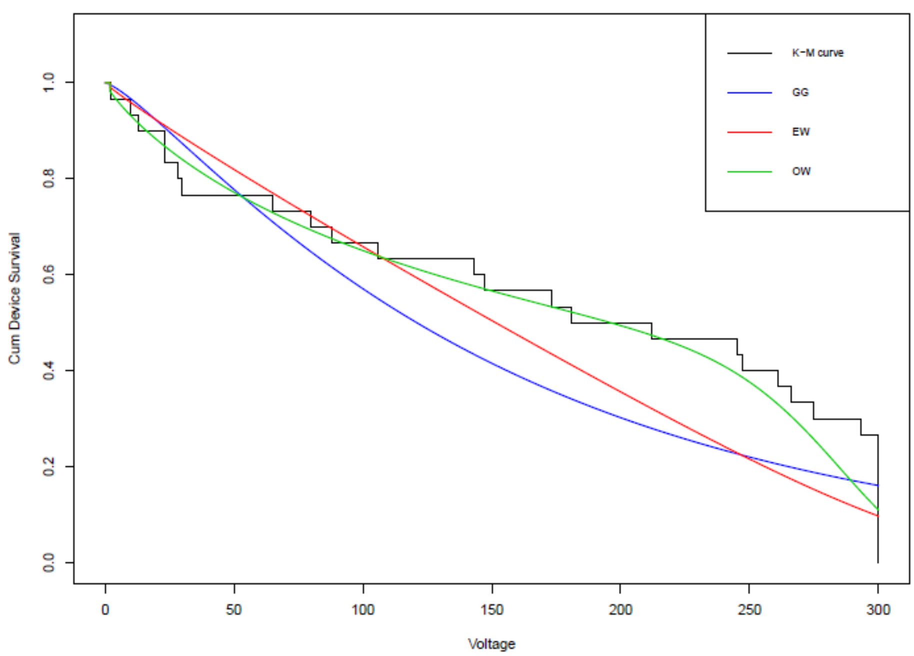


FIGURE 4: Fitted survival curves of the GG(blue), EW(red), and OW(green) distributions along with Kaplan-Meier(K-M) curve for voltage data.

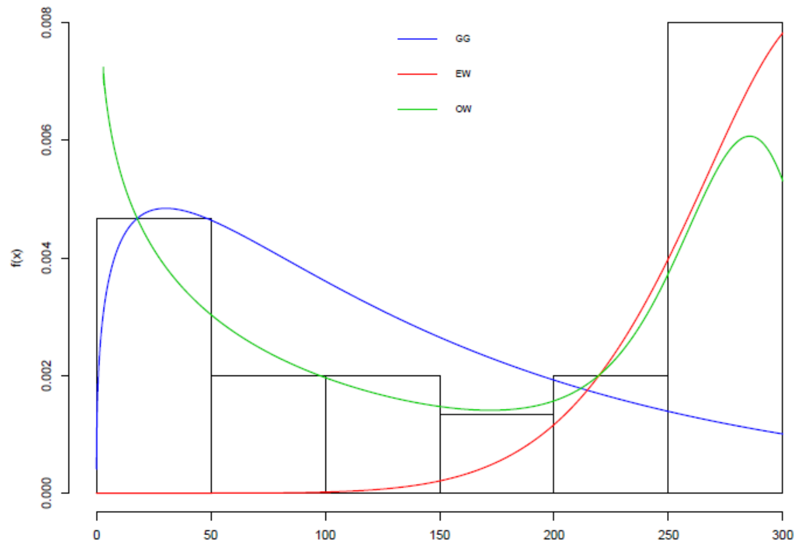


FIGURE 5: Fitted pdfs of the GG(blue), EW(red), and OW(green) distributions for voltage data.

Figure 4 shows the fitted survival curves of the GG, EW, and OW distributions, along with the Kaplan-Meier curve illustrating the effects of voltage in the survival of the devices. It is clear from Figure 4 that the OW distribution provides a better fit compared to the GG and EW distributions. Thus, the OW survival function provides a good parametric estimate for the Kaplan-Meier curve for modeling the voltage data. Figure 5 shows the pdfs of the GG, EW, and OW distributions fitted to the histogram of the voltage data. As shown in Figure 5, the OW distribution provides a better fit for the data since its pdf is able to fit the bath-tub shape of the histogram. In addition, Figure 6 shows the quantile-quantile plots for the GG, EW, and OW distributions, each plotted against the ordered observations. The p*th* quantile $\hat{Q}(p)$ was estimated from the p*th* quantile of the fitted distribution and $p = (r - 0.5)/n$, $r = 1, \dots, n$. The q-q plot shows that the OW distributions provides a better fit for the data, compared to the GG and EW distributions. Results from this graphical analysis agree with the results obtained in Table 2. Therefore, we conclude that the performance of the GG, EW, and OW distributions in fitting the voltage data is significantly different.

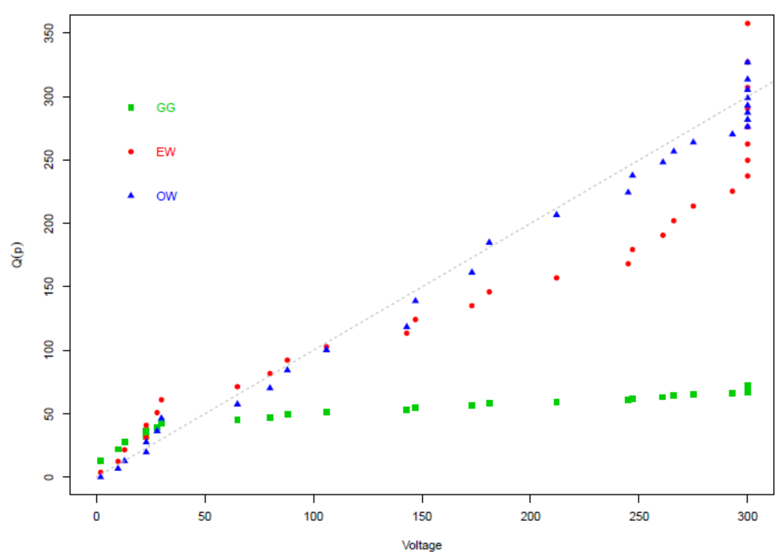


FIGURE 6: Q-Q plots of voltage data.

5. GENERALIZATION OF THE THREE WEIBULL EXTENSIONS

Motivated by the results in Section 3 and 4, we developed the following six-parameter generalized distribution function introduced by [1] which is an extension of the GG, EW, and OW distributions.

$$F\,(x;\,\lambda,\beta,\theta,\delta,\gamma,k) \;=\; [\Gamma\{log\;(1\;+\;\delta[exp\{(x/\theta)^{\lambda}\}\;-\;1]^{\beta}),k\}]^{\gamma} \tag{1}$$

where x > 0; θ > 0, γ > 0, δ > 0, k > 0, λβ > 0, $\Gamma(t,k) \;=\; (1/\Gamma(k))\int_0^t x^{k-1}exp(-x)dx, k > 0$, and $\Gamma(k)$ is the complete gamma function defined by $\Gamma(k) \;= \int_0^{\infty} t^{k-1}exp(-t)dt$. Submodels of (1) which have been studied in literature include the GG distribution (when δ = β = γ = 1), EW distribution (when δ = β = k = 1), and OW distribution (when γ = δ = k = 1). A four-parameter submodel of (1), the exponentiated Odd Webull distribution (EOW) with δ = k = 1 was then developed and studied in [1]. The EOW distribution has applications in modeling lifetime data, as illustrated in [1].

6. CONCLUSIONS

Our results showed that the performance of the OW differs significantly from either GG or EW distribution based on the tests considered. In addition, the OW distribution provides a better three-parameter alternative to the GG and EW distributions because of its hazard flexibility. Our analysis was an improvement to work previously done in literature since we used more general criteria for comparing the three distributions. From this study, we conclude that extensions of the GG, EW, and OW distributions may be developed to improve flexibility in the distribution and thus improve the fitting of data as done in [1]. This is necessary since recent technological advancements have resulted in more complex datasets which require more flexible distributions to provide a good fit to the data. As future work, additional extensions or generalizations of the GG, EW, and OW distributions will be considered using existing transformation methods in literature. Properties for other new submodels of the six-parameter generalized Weibull distribution in equation (1) will also be studied.

7. REFERENCES

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