On Fractional Fourier Transform Moments Based On Ambiguity Function

Sedigheh Ghofrani s_ghofrani@azad.ac.ir
Electrical Engineering Department, Islamic
Azad University, South Tehran Branch
Tehran, 15317, Iran

Abstract

The fractional Fourier transform can be considered as a rotated standard Fourier transform in general and its benefit in signal processing is growing to be known more. Noise removing is one application that fractional Fourier transform can do well if the signal dilation is perfectly known. In this paper, we have computed the first and second order of moments of fractional Fourier transform according to the ambiguity function exactly. In addition we have derived some relations between time and spectral moments with those obtained in fractional domain. We will prove that the first moment in fractional Fourier transform can also be considered as a rotated the time and frequency gravity in general. For more satisfaction, we choose five different types signals and obtain analytically their fractional Fourier transform and the first and second-order moments in time and frequency and fractional domains as well.

Keywords: Fractional Fourier Transform, Moments, Ambiguity Function.

1. INTRODUCTION

The uncertainty principle is a fundamental result in signal analysis. It is often called the duration-bandwidth theorem, which is perhaps more appropriate and descriptive for signals. Given a signal $x(t)$ and its Fourier transform (FT), $X(\omega)$, whenever we want to know the time or frequency-bandwidth, they can be calculated by:

$$\Delta_t = \sqrt{<t^2> - <t>^2} \qquad ; \qquad \Delta_\omega = \sqrt{<\omega^2> - <\omega>^2} \tag{1}$$

where:

$$<t^n> = \int_{-\infty}^{+\infty} t^n \left|x(t)\right|^2 dt \quad n \in Z^+ \qquad ; \qquad <\omega^n> = \int_{-\infty}^{+\infty} \omega^n \left|X(\omega)\right|^2 d\omega \quad n \in Z^+ \tag{2}$$

$$X(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \qquad ; \qquad x(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} X(\omega) e^{j\omega t} d\omega \tag{3}$$

In terms of these quantities, the standard uncertainty principle is $\Delta_t \Delta_\omega \geq \frac{1}{2}$. We notify that the spectral central moments can also be obtained using the time domain signal as:

$$<\omega^n> = \int_{-\infty}^{+\infty} x^*(t) \left(\frac{1}{j}\frac{d}{dt}\right)^n x(t) dt \quad n \in Z^+ \tag{4}$$

The uncertainty principle arises, because $x(t)$ and $X(\omega)$ are not arbitrary functions but are a FT pair. A proper interpretation of this result is that a signal cannot be both narrowband and short duration, since the variances of FT pairs cannot both be made arbitrarily small.

The FT is undoubtedly one of the most valuable and frequently used tools in theoretical and applied mathematics as well as signal processing and analysis. A generalization of FT, the fractional FT was first introduced from the mathematics aspect by Namis [1] and then considered by McBride [2]. They tried to make the theory of the fractional FT unambiguous and expressed the fractional FT in integral form similar to what now a day use. Fractional FT has been introduced in signal processing for the first time in [3]. It explored some relationship between the best well known time frequency distribution, Wigner Ville, and the fractional FT as well. Fractional FT of some functions in addition to its properties was given in [1], [3]-[5]. Specific features of the fractional FT for periodic signals were considered in [6]. Generally, in every area where FT and frequency domain concepts are used, there exists the potential for generalization and implementation by using fractional FT. In most of the signal processing applications, the signal which is to be recovered is degraded by additive noise. The concept of filtering in fractional Fourier domain is being realized. Some researchers noticed that signals with significant overlap in both time and frequency domain may have little or no overlap in a fractional Fourier domain [5], [7]. Filtering in a single time domain or in a single frequency domain has recently been generalized to filtering in a single fractional Fourier domain. They [8] further generalized the concept of signal fractional Fourier domain filtering to repeat filtering in consecutive fractional Fourier domains. A methodology for on-line tuning of transition bandwidth of windowed based FIR filters using fractional FT was proposed in [9]. The Fractional FT can be interpreted as decomposition of a signal in terms of chirps. In [10], an adaptive fractional Fourier domain filtering scheme in the presence of linear frequency modulated type noise was considered.

In this paper, we briefly introduce the fractional FT and a number of its properties and then present some new results; the fractional moments are independently derived and some relationship between moments belong to ordinary and fractional plane are proved; in addition example of fractional FT's of some new and useful signals are obtained and their moments are directly determined.

This paper is organized as follows. In section 2, we present the fractional FT and list some of its properties. In section 3, we derive the first and second fractional FT moments according to ambiguity function (AF) and find some relations among time-frequency and fractional moments respectively. In section 4, we obtain the fractional FT of some signals those can be used as an additive noise model, and obtain the first and second their fractional moments. Finally section 5 concludes the paper.

Note on the Formalism: we will represent by "j" the imaginary unit ($\sqrt{-1}$) and by a superscript asterisk '*' the complex conjugate operation.

2. FRACTIONAL FOURIER TRANSFORM

In the mathematics literature, the concept of fractional order FT was proposed some years ago [1], [2], [5]. The ordinary FT being a transform of order 1, and the signal in time is of order zero. The fractional FT depends on a parameter α and can be interpreted as a rotation by an angle α in the time- frequency plane. The relationship between fractional FT order and angle is given by $\alpha = a\frac{\pi}{2}$. This section gives a compact review of the theory of fractional FT and some properties that will be used throughout this paper. The fractional FT of function $x(t)$ can be written in the form:

$$X_\alpha(u) = \int_{-\infty}^{+\infty} x(t) K_\alpha(t,u) dt \tag{5}$$

the kernel $K_\alpha(t,u)$ is given by $K_\alpha(t,u) = \sqrt{\frac{1-j\cot\alpha}{2\pi}} e^{j\frac{t^2+u^2}{2}\cot\alpha - jut\csc\alpha}$. The parameter α is continuous and interpreted as a rotation angle in the phase plane. When α increases from 0 to $\frac{\pi}{2}$, the fractional FT produce a continuous transformation of a signal to its Fourier image. If α or $\alpha+\pi$ is a multiple of 2π, the kernel reduces to $\delta(t-u)$ or $\delta(t+u)$ respectively. We also note that for $\alpha = \frac{\pi}{2}$, the kernel coincide with the kernel of the ordinary FT. In summary, the fractional FT is a linear transform, and continuous in the angle α, which satisfies the basic conditions for being interpretable as a rotation in the time- frequency plane [3]. Fractional FT is the energy-preserving transform [3], it means:

$$\int_{-\infty}^{+\infty} |x(t)|^2 \, dt = \int_{-\infty}^{+\infty} |X_\alpha(u)|^2 du \tag{6}$$

Due to the energy-preserving property of the FT, the squared magnitude of the FT of a signal $|X(\omega)|^2$ is often called the energy spectrum of the signal and is interpreted as the distribution of the signal's energy among the different frequencies. As the fractional FT is also energy conservative, $|X_\alpha(u)|^2$ is named as the fractional energy spectrum of the signal x(t), with angle α.

In time-frequency representations, one normally uses a plane with two orthogonal axes corresponding to time and frequency respectively, (Fig. 1).

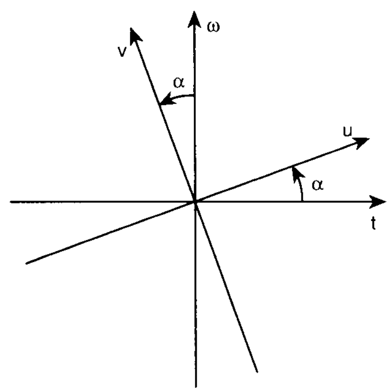

FIGURE 1: time- frequency plane and a set of coordinates (u, v) rotated by an angle α relative to the original coordinates (t, ω).

A signal represented along the frequency axis is the FT of the signal representation $x(t)$ along the time axis. It can also be represented along an axis making some angle α with the time axis. Along this axis, we define the fractional FT of $x(t)$ at angle α defined as the linear integral transform (Eq. 5). It is easy to prove that pairs (t, ω) and (u, v) corresponding to an axis rotation by:

$$\begin{pmatrix} t \\ \omega \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \tag{7}$$

Although many properties were known for fractional FT, it is convenient to include in this preliminary section three results which will be useful later on. Now according to $x(t) \overset{FrFT}{\Longleftrightarrow} X_\alpha(u)$, we denote these properties, they are named shift, modulation, and multiplication as follows:

$$x(t-\tau) \overset{FrFT}{\Longleftrightarrow} e^{j\frac{\tau^2}{2}\sin\alpha\cos\alpha - ju\tau\sin\alpha} X_\alpha(u - \tau\cos\alpha) \tag{8}$$

$$x(t)e^{-j\theta t} \overset{FrFT}{\Longleftrightarrow} e^{-j\frac{\theta^2}{2}\sin\alpha\cos\alpha - ju\theta\cos\alpha} X_\alpha(u + \theta\sin\alpha) \tag{9}$$

$$tx(t) \overset{FrFT}{\Longleftrightarrow} u\cos\alpha \, X_\alpha(u) + j\sin\alpha \, X'_\alpha(u) \tag{10}$$

3. FRACTIONAL MOMENTS BASED ON AMBIGUITY FUNCTION

We suppose that an optimal fractional domain corresponds to minimum signal width. Calculation of this moment can be done analytically, based on using the AF which can be interpreted as a joint time- frequency auto correlation function. In this paper, based on connection between the AF and the fractional FT, we derive the fractional moments, though the first and second moments were obtained in [11] and [12] before. These moments are related to the fractional energy spectra and therefore can be easily measured for example in signal analysis. The AF of a signal $x(t)$ is defined as [13]:

$$AF_x(\theta,\tau)=\int_{-\infty}^{+\infty} x(t+\frac{\tau}{2})x^*(t-\frac{\tau}{2})e^{-j\theta t}dt=\int_{-\infty}^{+\infty} X(\omega+\frac{\theta}{2})X^*(\omega-\frac{\theta}{2})e^{j\tau\omega}d\omega \qquad (11)$$

It is easy to show that:

$$AF_x(\theta,0)=\int_{-\infty}^{+\infty} x(t)x^*(t)e^{-j\theta t}dt \qquad <t^n>=\frac{1}{(-j)^n}\cdot\left.\frac{\partial^n AF_x(\theta,0)}{\partial\theta^n}\right|_{\theta=0} \qquad n\in Z^+ \qquad (12)$$

$$AF_x(0,\tau)=\int_{-\infty}^{+\infty} X(\omega)X^*(\omega)e^{j\tau\omega}d\omega \qquad <\omega^n>=\frac{1}{(j)^n}\cdot\left.\frac{\partial^n AF_x(0,\tau)}{\partial\tau^n}\right|_{\tau=0} \qquad n\in Z^+ \qquad (13)$$

Before starting to derive the first and second-order moments in fractional FT based on the moments in time and frequency, we recall that as fractional FT is a linear transform and energy conservative, so in general the fractional moments can be considered as:

$$<u^n>=\int_{-\infty}^{+\infty} u^n\,|X_\alpha(u)|^2du \qquad n\in Z^+ \qquad (14)$$

Now according to the fractional FT definition (Eq. 5), and shift and modulation properties (Eqs. 9 and 10), we rewrite the AF as follows:

$$AF_x(\theta,\tau)=e^{j\frac{\theta\tau}{2}}e^{j\frac{u^2}{2\sin\alpha\cos\alpha}}\int_{-\infty}^{+\infty} e^{-j\frac{\sin\alpha\cos\alpha}{2}(\tau-\frac{u}{\cos\alpha})^2}X_\alpha^*(u-\tau\cos\alpha)\cdot e^{-j\frac{\sin\alpha\cos\alpha}{2}(\theta+\frac{u}{\sin\alpha})^2}X_\alpha(u+\theta\sin\alpha)du \qquad (15)$$

3.1 Time Moments

Although it takes really long analytic computation, we try to obtain the first and second-order moments belong to time according to Eq. (12) and by using Eq. (15) as follows:

$$AF_x(\theta,0)=e^{j\frac{1}{2\tan\alpha}u^2}\int_{-\infty}^{+\infty} X_\alpha^*(u)\cdot e^{-j\frac{\sin\alpha\cos\alpha}{2}(\theta+\frac{u}{\sin\alpha})^2}X_\alpha(u+\theta\sin\alpha)du \qquad (16)$$

Easy using equation (16), and (11) show the fractional FT is energy conservative or unique signal has unique fractional FT and so the transform is reversible ($E_x=AF_x(0,0)=\int_{-\infty}^{+\infty}|x(t)|^2\,dt=\int_{-\infty}^{+\infty}|X_\alpha(u)|^2du$). We consider the signal energy is 1 ($E_x=1$). Now we determine the first derivative in order to determine the first order moment in time domain:

$$\left.\frac{\partial AF_x(\theta,0)}{\partial\theta}\right|_{\theta=0}=\int_{-\infty}^{+\infty} X_\alpha^*(u)\cdot\left.\frac{\partial X_\alpha(u+\theta\sin\alpha)}{\partial\theta}\right|_{\theta=0}du+\int_{-\infty}^{+\infty}(-j\cos\alpha)u\,|X_\alpha(u)|^2du \qquad (17)$$

$$<t>=j\int_{-\infty}^{+\infty} X_\alpha^*(u)\cdot\left.\frac{\partial X_\alpha(u+\theta\sin\alpha)}{\partial\theta}\right|_{\theta=0}du+\cos\alpha<u> \qquad (18)$$

Similarly, the second-order moment in time domain can be obtained by the second derivative of AF as:

$$\left.\frac{\partial^2 AF_x(\theta,0)}{\partial\theta^2}\right|_{\theta=0} = \int_{-\infty}^{+\infty} X_\alpha^*(u)\cdot\left.\frac{\partial^2 X_\alpha(u+\theta\sin\alpha)}{\partial\theta^2}\right|_{\theta=0} du - 2j(\cos\alpha)\int_{-\infty}^{+\infty} uX_\alpha^*(u)\cdot\left.\frac{\partial X_\alpha(u+\theta\sin\alpha)}{\partial\theta}\right|_{\theta=0} du$$
$$+\int_{-\infty}^{+\infty}(-j\sin\alpha\cos\alpha)\,|\,X_\alpha(u)\,|^2 du + \int_{-\infty}^{+\infty}(-j\cos\alpha)^2 u^2\,|\,X_\alpha(u)\,|^2 du \tag{19}$$

$$< t^2 >= -\int_{-\infty}^{+\infty} X_\alpha^*(u)\left.\frac{\partial^2 X_\alpha(u+\theta\sin\alpha)}{\partial\theta^2}\right|_{\theta=0} du + 2j(\cos\alpha)\int_{-\infty}^{+\infty} uX_\alpha^*(u)\left.\frac{\partial X_\alpha(u+\theta\sin\alpha)}{\partial\theta}\right|_{\theta=0} du$$
$$+ j\frac{\sin 2\alpha}{2} + \cos^2\alpha < u^2 > \tag{20}$$

Now we should simplify the derived equations for the first and second-order moments in time domain. Using Eqs. (5) and (15) for fractional FT definition, it is not too hard to prove the following relationship:

$$\left.\frac{\partial X_\alpha(u+\theta\sin\alpha)}{\partial\theta}\right|_{\theta=0} = \sin\alpha\frac{\partial X_\alpha(u)}{\partial u} \qquad \left.\frac{\partial^2 X_\alpha(u+\theta\sin\alpha)}{\partial\theta^2}\right|_{\theta=0} = \sin^2\alpha\frac{\partial^2 X_\alpha(u)}{\partial u^2} \tag{21}$$

Thereby, we rewrite the first and second order moments:

$$< t >= j\sin\alpha\int_{-\infty}^{+\infty} X_\alpha^*(u)\cdot\frac{\partial X_\alpha(u)}{\partial u} du + \cos\alpha < u > \tag{22}$$

$$< t^2 >= -\sin^2\alpha\int_{-\infty}^{+\infty} X_\alpha^*(u)\cdot\frac{\partial^2 X_\alpha(u)}{\partial u^2} du + j\sin 2\alpha\int_{-\infty}^{+\infty} uX_\alpha^*(u)\cdot\frac{\partial X_\alpha(u)}{\partial u} du + j\frac{\sin 2\alpha}{2} + \cos^2\alpha < u^2 > \tag{23}$$

As u and v are orthogonal axes (Fig. 1), we can obtain moment in v domain by using signal in u domain as:

$$< v^n >= \int_{-\infty}^{+\infty} X_\alpha^*(u)\left(\frac{1}{j}\frac{\partial}{\partial u}\right)^n X_\alpha(u) du \quad n\in Z^+ \tag{24}$$

Now the first and second order moments for time domain and then duration are obtained according to the fractional moments:

$$< t >= -\sin\alpha < v > + \cos\alpha < u > \tag{25}$$

$$< t^2 >= \sin^2\alpha < v^2 > + j\sin 2\alpha\int_{-\infty}^{+\infty} uX_\alpha^*(u)\cdot\frac{\partial X_\alpha(u)}{\partial u} du + j\frac{\sin 2\alpha}{2} + \cos^2\alpha < u^2 > \tag{26}$$

$$\Delta_t^2 =< t^2 > - < t >^2 = \sin^2\alpha\,\Delta_v^2 + \cos^2\alpha\,\Delta_u^2 + j\frac{\sin 2\alpha}{2} + \sin 2\alpha < v >< u > + j\sin 2\alpha\int_{-\infty}^{+\infty} uX_\alpha^*(u)\cdot\frac{\partial X_\alpha(u)}{\partial u} du \tag{27}$$

where Δ_u^2 and Δ_v^2 refer to signal dilation in fractional plane.

3.2 Frequency Moments

Exactly the same algebra is used in order to obtain frequency moments. By notifying Eq. (13) and usind Eq. (15), we write:

$$AF_x(0,\tau) = e^{j\frac{\tan\alpha}{2}u^2}\int_{-\infty}^{+\infty} X_\alpha(u)\cdot e^{-j\frac{\sin\alpha\cos\alpha}{2}(\tau-\frac{u}{\cos\alpha})^2} X_\alpha^*(u-\tau\cos\alpha) du \tag{28}$$

$$\left.\frac{\partial AF_x(0,\tau)}{\partial\tau}\right|_{\tau=0} = \int_{-\infty}^{+\infty} X_\alpha(u)\cdot\left.\frac{\partial X_\alpha^*(u-\tau\cos\alpha)}{\partial\tau}\right|_{\tau=0} du + \int_{-\infty}^{+\infty}(j\sin\alpha)u\,|\,X_\alpha(u)\,|^2 du \tag{29}$$

$$< \omega > = -j \int_{-\infty}^{+\infty} X_\alpha(u) \cdot \left. \frac{\partial X_\alpha^*(u - \tau \cos\alpha)}{\partial \tau} \right|_{\tau=0} du + \sin\alpha < u > \tag{30}$$

As we consider the signal energy equal to 1 then the spectral second order moment is:

$$\begin{aligned} \left. \frac{\partial^2 AF_x(0,\tau)}{\partial \tau^2} \right|_{\tau=0} &= \int_{-\infty}^{+\infty} X_\alpha(u) \cdot \left. \frac{\partial^2 X_\alpha^*(u - \tau\cos\alpha)}{\partial \tau^2} \right|_{\tau=0} du \\ &+ 2j\sin\alpha \int_{-\infty}^{+\infty} u X_\alpha(u) \cdot \left. \frac{\partial X_\alpha^*(u - \tau\cos\alpha)}{\partial \tau} \right|_{\tau=0} du - j\frac{\sin 2\alpha}{2} - \sin^2\alpha < u^2 > \end{aligned} \tag{31}$$

$$\begin{aligned} < \omega^2 > &= -\int_{-\infty}^{+\infty} X_\alpha(u) \cdot \left. \frac{\partial^2 X_\alpha^*(u - \tau\cos\alpha)}{\partial \tau^2} \right|_{\tau=0} du \\ &- 2j\sin\alpha \int_{-\infty}^{+\infty} u X_\alpha(u) \cdot \left. \frac{\partial X_\alpha^*(u - \tau\cos\alpha)}{\partial \tau} \right|_{\tau=0} du + j\frac{\sin 2\alpha}{2} + \sin^2\alpha < u^2 > \end{aligned} \tag{32}$$

In order to simplify the derived equations, the following relations by employing Eqs. (5) and (15) are determined:

$$\left. \frac{\partial X_\alpha^*(u - \tau\cos\alpha)}{\partial \tau} \right|_{\tau=0} = -\cos\alpha \frac{\partial X_\alpha^*(u)}{\partial u} \qquad \left. \frac{\partial^2 X_\alpha^*(u - \tau\cos\alpha)}{\partial \tau^2} \right|_{\tau=0} = \cos^2\alpha \frac{\partial^2 X_\alpha^*(u)}{\partial u^2} \tag{33}$$

And now, the first and second spectral moments and also signal bandwidth are written as follows:

$$< \omega > = \cos\alpha < v > + \sin\alpha < u > \tag{34}$$

$$< \omega^2 > = \sin^2\alpha < u^2 > + \cos^2\alpha < v^2 > + j\frac{\sin 2\alpha}{2} + j\sin 2\alpha \left(\int_{-\infty}^{+\infty} u X_\alpha^*(u) \cdot \frac{\partial X_\alpha(u)}{\partial \tau} du \right)^* \tag{35}$$

$$\Delta_\omega^2 = < \omega^2 > - < \omega >^2 = \sin^2\alpha\, \Delta_u^2 + \cos^2\alpha\, \Delta_v^2 + j\frac{\sin 2\alpha}{2} - \sin 2\alpha < u >< v > + j\sin 2\alpha \left(\int_{-\infty}^{+\infty} u X_\alpha^*(u) \cdot \frac{\partial X_\alpha(u)}{\partial \tau} du \right)^* \tag{36}$$

In order to make the derived equations more readable, we define the first and second order moments and signal dilation according to their corresponding plane. They are $m_0 = < t >$; $m_{\frac{\pi}{2}} = < \omega >$; $m_\alpha = < u >$; $m_{\alpha+\frac{\pi}{2}} = < v >$, and $w_0 = < t^2 >$; $w_{\frac{\pi}{2}} = < \omega^2 >$; $w_\alpha = < u^2 >$; $w_{\alpha+\frac{\pi}{2}} = < v^2 >$, and $\mu_0 = w_0 - m_0^2$; $\mu_{\frac{\pi}{2}} = w_{\frac{\pi}{2}} - m_{\frac{\pi}{2}}^2$; $\mu_\alpha = w_\alpha - m_\alpha^2$; $\mu_{\alpha+\frac{\pi}{2}} = w_{\alpha+\frac{\pi}{2}} - m_{\alpha+\frac{\pi}{2}}^2$.

Result1: according to the derived equations (25) and (34), and using the above definitions, we have:

$$\begin{pmatrix} m_0 \\ m_{\frac{\pi}{2}} \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} m_\alpha \\ m_{\alpha+\frac{\pi}{2}} \end{pmatrix} \tag{37}$$

As rotation is true for pairs (t,ω) and (u,v), (Eq. 8), obviously it is also true for the first moments in original plane. This result emphasize on why fractional FT is considered as a rotation operator. The first order moment, $m_\alpha = < u >$, in a fractional domain defined by an arbitrary angle α can be calculated from the relationship $m_\alpha = \cos\alpha\, m_0 + \sin\alpha\, m_{\frac{\pi}{2}}$.

Result 2: Taking into account Eqs. (25), (26), (34), and (35), we conclude the following relationships:

$$m_0^2 + m_{\frac{\pi}{2}}^2 = m_\alpha^2 + m_{\alpha+\frac{\pi}{2}}^2 \quad ; \quad w_0 + w_{\frac{\pi}{2}} = w_{\alpha+\frac{\pi}{2}} + w_\alpha \quad ; \quad \mu_0 + \mu_{\frac{\pi}{2}} = \mu_\alpha + \mu_{\alpha+\frac{\pi}{2}} \tag{38}$$

According to what are derived, we can say the first, and the second moments as well as dilation are rotate invariant.

Result 3: At first we notice to Eqs. (27) and (36), that duration in time or bandwidth in frequency domain should be real positive value because of physical interpretation, so the imaginary parts in two above referred equations are to be considered equal zero. In addition if signal in fractional domain supposed to be real, then we have:

$$\mu_0 = \sin^2\alpha\, \mu_{\alpha+\frac{\pi}{2}} + \cos^2\alpha\, \mu_\alpha + \sin 2\alpha\, m_\alpha m_{\alpha+\frac{\pi}{2}} \tag{39}$$

$$\mu_{\frac{\pi}{2}} = \cos^2\alpha\, \mu_{\alpha+\frac{\pi}{2}} + \sin^2\alpha\, \mu_\alpha - \sin 2\alpha\, m_\alpha m_{\alpha+\frac{\pi}{2}} \tag{40}$$

So in a fractional domain defined by an arbitrary angle α , the signal dilation can be computed by duration in time, bandwidth in frequency and the first order moments.

4. DIFFERENT SIGNALS

Fractional FT of a number of common signals such as $\exp(-t^2/2)$, $\delta(t)$, and e^{jkt} were computed before [1]. It was proved that fractional FT also exist for certain functions which are not square integrable (for example: $1, t, t^2, \text{etc}$) [1] (as in Z transform using r causes having this feature, here being α causes this effect). Fractional FT has attracted a great attention. Some researchers try to discover its features more [6], and some try to use it in application. Conventionally, the filtering systems are based on the FT, though the frequency of the noise and that of the signal usually overlap with each other, so it is very difficult to filter the noise completely. So it may conclude that filtering in the optimal fractional domain is significantly better than filtering in the conventional frequency domain. Fractional Fourier domain filtering in a single domain is particularly advantageous when the distortion or noise is of a chirped nature [7], [10], [14]. For further application of the fractional FT analysis, it is important to study its effects on different types of signals. It was suggested that instead of filtering in time or frequency, it can be done better in rotated domain where the signal spreading is low. It means that obtaining the central moments and explore their behavior are important topic for design an optimum filter in rotated domain or fractional FT. In this section, we will obtain the fractional FT for five different type functions which can be considered as a model for additive noise. We also compute the corresponding first and second fractional order moments and derived some relations among time, frequency and characteristics belong to rotated coordinates as well. At the end, we notify that as the energy of these five signals are not equal to 1, we divide the calculated moments by signal energy.

4.1 Gaussian Function

We consider $x(t) = e^{-\frac{t^2}{2\sigma^2}}$ as a Gaussian function, the signal's energy is equal to $E_x = \sqrt{\pi}\sigma$ and the standard FT is, $X(j\omega) = \sigma \cdot e^{-\frac{\sigma^2}{2}\omega^2}$. Although, the fractional FT of Gaussian function was computed before, here we determine it again:

$$X_\alpha(u) = \sqrt{\frac{1 - j\cot\alpha}{\frac{1}{\sigma^2} - j\cot\alpha}} \cdot \exp\left(\frac{-u^2}{2} \cdot \frac{1 - \frac{j}{\sigma^2}\cot\alpha}{\frac{1}{\sigma^2} - j\cot\alpha}\right) \tag{41}$$

Obviously, it is easy to show that $X_\alpha(u)\big|_{\alpha=\frac{\pi}{2}} = X(j\omega)$, this result prove the computed procedure has done correctly. The central moments in time, frequency, and fractional domain are written in Tabel 1.

$\Vert x(t)\Vert^2 = e^{-\frac{t^2}{\sigma^2}}$	$m_0 = 0$	$w_0 = \frac{\sigma^2}{2}$	$\mu_0 = \frac{\sigma^2}{2}$
$\Vert X(j\omega)\Vert^2 = \sigma^2 e^{-\sigma^2\omega^2}$	$m_{\frac{\pi}{2}} = 0$	$w_{\frac{\pi}{2}} = \frac{1}{2\sigma^2}$	$\mu_{\frac{\pi}{2}} = \frac{1}{2\sigma^2}$
$\Vert X_\alpha(u)\Vert^2 = \frac{\sigma}{k}\cdot\exp(-\frac{u^2}{k^2})$; $k^2 = \frac{\sin^2\alpha}{\sigma^2} + \sigma^2\cos^2\alpha$	$m_\alpha = 0$	$w_\alpha = \frac{k^2}{2}$	$\mu_\alpha = \frac{k^2}{2}$

TABLE 1: The central moments of Gaussian function.

We see that Eq. (37) is satisfied. On the other hand, it was proved in [15], for any real valued signal inequality, $\mu_\alpha\mu_\beta \geq [\mu_0\cos\alpha\cos\beta + \frac{\sin\alpha\sin\beta}{4\mu_0}]^2 + (\frac{\sin(\alpha-\beta)}{2})^2$, is always satisfied. Now for Gaussian signal, we have $\mu_\alpha\mu_\beta = \frac{1}{4}[\sigma^2\cos\alpha\cos\beta + \frac{\sin\alpha\sin\beta}{\sigma^2}]^2 + (\frac{\sin(\alpha-\beta)}{2})^2$. It means that Gaussian function has the least dilation not only in time and frequency domain but also in fractional domain among all different signals. Now, if we compute $\frac{\partial\mu_\alpha}{\partial\alpha}$ for Gaussian function, we see that for $|\sigma|<1$, the least dilation happens in time and for $|\sigma|>1$, the least dilation happens at an angle $\alpha = \frac{\pi}{2}$ in frequency domain. So considering Gaussian as an additive noise, it is better to perform filtering in time or frequency not in fractional domain.

4.2 Laplace Function

The laplace function is $x(t) = e^{-b|t|}; b > 0$ and its energy is equal to $E_x = \frac{1}{b}$. The classic FT is, $X(j\omega) = \frac{2b}{\sqrt{2\pi}(b^2+\omega^2)}$, and the fractional FT is obtained:

$$X_\alpha(u) = \frac{\sqrt{1+j\tan\alpha}}{2}\cdot\exp(\frac{j}{2\cot\alpha}(b^2-u^2))\cdot(\exp(\frac{u}{\cos\alpha}\cdot b) + \exp(\frac{-u}{\cos\alpha}\cdot b)) \tag{42}$$

According to the derived Eq (42), we suggest the fractional FT of Laplace function is written as follows:

$$X_\alpha(u) = \sqrt{1+j\tan\alpha}\cdot\exp(\frac{j}{2\cot\alpha}(b^2-u^2))\cdot\exp(\frac{-|u|}{\cos\alpha}\cdot b) \tag{43}$$

The central moments in time, frequency, and fractional domain are written in Tabel 2.

$\Vert x(t)\Vert^2 = e^{-2b\|t\|}$	$m_0 = 0$	$w_0 = \frac{1}{2b^2}$	$\mu_0 = \frac{1}{2b^2}$
$\Vert X(j\omega)\Vert^2 = \frac{1}{2\pi}\frac{4b^2}{(b^2+\omega^2)^2}$	$m_{\frac{\pi}{2}} = 0$	$w_{\frac{\pi}{2}} = b^2$	$\mu_{\frac{\pi}{2}} = b^2$
$\Vert X_\alpha(u)\Vert^2 = \frac{1}{\|\cos\alpha\|}\cdot\exp(\frac{-2\|u\|}{\cos\alpha}\cdot b)$	$m_\alpha = 0$	$w_\alpha = \frac{1}{2}(\frac{\cos\alpha}{b})^2$	$\mu_\alpha = \frac{1}{2}(\frac{\cos\alpha}{b})^2$

TABLE 2: The central moments of Laplace function.

Obtaining $\frac{\partial\mu_\alpha}{\partial\alpha}$, we conclude Laplace function has the least dilation in time domain.

4.3 One Sided Gaussian Function

One sided Gaussian function is $x(t)=e^{-\frac{t^2}{2\sigma^2}}u(t)$, and the signal's energy is equal to $E_x=\frac{1}{2}\sqrt{\pi}\sigma$. Although it is really simple, we obtain the FT, $X(j\omega)=\frac{\sigma}{2}e^{-\frac{\sigma^2}{2}\omega^2}$, and the fractional FT as:

$$X_\alpha(u)=\frac{1}{2}\sqrt{\frac{1-j\cot\alpha}{\frac{1}{\sigma^2}-j\cot\alpha}}\cdot\exp\left(\frac{-u^2}{2}\cdot\frac{1-\frac{j}{\sigma^2}\cot\alpha}{\frac{1}{\sigma^2}-j\cot\alpha}\right) \tag{44}$$

As it is seen, $X_\alpha(u)\big|_{\alpha=\frac{\pi}{2}}=X(j\omega)$, and this result show the derived fractional FT for one sided Gaussian function is definitely correct. The central moments in time, frequency, and fractional domain are written in Tabel 3, the value of k is the same as what defined for Gaussian function.

$\lVert x(t)\rVert^2=e^{-\frac{t^2}{\sigma^2}}u(t)$	$m_0=\frac{\sigma}{\sqrt{\pi}}$	$w_0=\frac{\sigma^2}{2}$	$\mu_0=\sigma^2(\frac{1}{2}-\frac{1}{\pi})$
$\lVert X(j\omega)\rVert^2=\frac{\sigma^2}{4}e^{-\sigma^2\omega^2}$	$m_{\frac{\pi}{2}}=0$	$w_{\frac{\pi}{2}}=\frac{1}{4\sigma^2}$	$\mu_{\frac{\pi}{2}}=\frac{1}{4\sigma^2}$
$\lVert X_\alpha(u)\rVert^2=\frac{\sigma}{4k}\cdot\exp(-\frac{u^2}{k^2})$	$m_\alpha=0$	$w_\alpha=\frac{1}{4}(\frac{1}{\sigma^2}\sin^2\alpha+\sigma^2\cos^2\alpha)$	$\mu_\alpha=\frac{1}{4}(\frac{1}{\sigma^2}\sin^2\alpha+\sigma^2\cos^2\alpha)$

TABLE 3: The central moments of one sided Gaussian function.

4.4 Rayleigh Function

Rayleigh function, $x(t)=te^{-\frac{t^2}{2\sigma^2}}\mathrm{u}(t)$, is known especially in wireless communication. Its energy is equal to $E_x=\sqrt{\pi}\frac{\sigma^3}{4}$, and the FT is $X(j\omega)=\frac{\sigma^2}{\sqrt{2\pi}}-\frac{\sigma^3}{2}\cdot j\omega e^{-\frac{\sigma^2}{2}\omega^2}$. Now we obtain the fractional FT for Rayleigh function:

$$\begin{aligned}X_\alpha(u)=&\sqrt{\frac{1-j\cot\alpha}{2\pi}}\cdot\frac{1}{\frac{1}{\sigma^2}-j\cot\alpha}\exp\left(\frac{u^2}{2}j\cot\alpha\right)\\&+\frac{1}{2}\frac{u}{j\sin\alpha}\cdot\frac{1}{\frac{1}{\sigma^2}-j\cot\alpha}\sqrt{\frac{1-j\cot\alpha}{\frac{1}{\sigma^2}-j\cot\alpha}}\cdot\exp\left(\frac{-u^2}{2}\cdot\frac{1-\frac{j}{\sigma^2}\cot\alpha}{\frac{1}{\sigma^2}-j\cot\alpha}\right)\end{aligned} \tag{45}$$

As it is seen, $X_\alpha(u)\big|_{\alpha=\frac{\pi}{2}}=X(j\omega)$, and this result show the derived fractional FT for Rayleigh function is correct. On the other hand, according to the property that described by Eq. (10), we are able to find the fractional FT of Rayleigh function by using the one sided Gaussian function as follow:

$$tx(t) = te^{-\frac{t^2}{2\sigma^2}}\,\mathrm{u}(t) \overset{FrFT}{\Longleftrightarrow} \frac{-ju}{\sin\alpha}\cdot\frac{1}{\frac{1}{\sigma^2}-j\cot\alpha}\cdot\frac{1}{2}\sqrt{\frac{1-j\cot\alpha}{\frac{1}{\sigma^2}-j\cot\alpha}}\cdot\exp\left(\frac{-u^2}{2}\cdot\frac{1-\frac{j}{\sigma^2}\cot\alpha}{\frac{1}{\sigma^2}-j\cot\alpha}\right) \quad (46)$$

It is seen that there is not constant term in (46). The time and the frequency moments of this function are written in Table 4, though because of complexity of the Eq. (45) we could not find the fractional moments analytically.

$\Vert x(t)\Vert^2 = t^2 e^{-\frac{t^2}{\sigma^2}} u(t)$	$m_0 = \frac{2\sigma}{\sqrt{\pi}}$	$w_0 = \frac{3\sigma^2}{2}$	$\mu_0 = \sigma^2(\frac{3}{2}-\frac{4}{\pi})$
$\Vert X(j\omega)\Vert^2 = \frac{\sigma^4}{2\pi}+\frac{\sigma^6}{4}\omega^2 e^{-\sigma^2\omega^2}$	$m_{\frac{\pi}{2}} = 0$	$w_{\frac{\pi}{2}} = \frac{3}{4\sigma^2}$	$\mu_{\frac{\pi}{2}} = \frac{3}{4\sigma^2}$

TABLE 4: The central moments of Rayleigh function.

Although we could not find m_α directly, according to the derived relationship in Eq. (37), it may conclude that $m_\alpha = \frac{2\sigma}{\sqrt{\pi}}\cos\alpha$.

4.5 One Sided Exponential Function

One sided exponential is $x(t) = b^t\,\mathrm{u}(t); 0 < b < 1$, and its energy is equal to $E_x = \frac{-1}{2\ln b}$. The standard FT is $X(j\omega) = \frac{1}{\sqrt{2\pi}}\cdot\frac{1}{-\ln b + j\omega}$ and the fractional FT is obtained as:

$$X_\alpha(u) = \sqrt{1+j\tan\alpha}\cdot\exp\left(-\sigma^2\frac{u}{\cos\alpha}\right)\cdot\exp\left(\frac{u^2-\sigma^4}{2j\cot\alpha}\right) \quad (47)$$

$\Vert x(t)\Vert^2 = b^{2t}\,\mathrm{u}(t)$	$m_0 = \frac{-1}{2\ln b}$	$w_0 = \frac{1}{2(\ln b)^2}$	$\mu_0 = \frac{1}{4(\ln b)^2}$
$\Vert X(j\omega)\Vert^2 = \frac{1}{2\pi}\cdot\frac{1}{(\ln b)^2+\omega^2}$	$m_{\frac{\pi}{2}} = 0$	$w_{\frac{\pi}{2}} = -\ln b$	$\mu_{\frac{\pi}{2}} = -\ln b$
$\Vert X_\alpha(u)\Vert^2 = \frac{-2\ln b}{\lvert\cos\alpha\rvert}\cdot\exp\left(-2\sigma^2\frac{u}{\cos\alpha}\right) \quad ; \quad u \ge 0$	$m_\alpha = \frac{\lvert\cos\alpha\rvert}{-2\ln b}$	$w_\alpha = \frac{2(\cos\alpha)^2}{(2\ln b)^2}$	$\mu_\alpha = \frac{(\cos\alpha)^2}{(2\ln b)^2}$

TABLE 5: The central moments of one sided exponential function.

It is seen that the signal has always the least dilation in time domain.

5. CONCLUSIONS

The fractional Fourier transform moments may be helpful in the search for the most appropriate fractional domain to perform a filtering operation; in the special case of noise that is equally distributed throughout the time-frequency plane, for instance, the fractional domain with the smallest signal width is then evidently the most preferred one. In this paper we have derived the new relations between central moments in time, frequency, and fractional domain by employing the ambiguity function. In addition, we have obtained the fractional Fourier transform and fractional moments for different signals directly. Thereby we conclude except chirp signal, there are many signals whose dilation are least in time or frequency, the original plane not rotated plane.

6. REFERENCES

[1] V. Namis, “The fractional order Fourier transform and its application to quantum mechanics,” Journal of Inst. Maths Applics. , 25, pp. 241- 265, 1980.

[2] A. C. McBride, and F. H. Kerr, “On Namias’s fractional Fourier transform,” IMA Journal of applied mathematics, 39, pp. 159-m175, 1987.

[3] L. B. Almedia, “The fractional Fourier transform and time- frequency representations,” IEEE Trans. On Signal Processing, Vol. 42, No. 11, pp. 3084- 3091, Nov. 1994.

[4] H. M. Ozaktas, N. Erkaya, and M. A. Kutay, “Effect of fractional Fourier transformation on time- frequency distributions belonging to the Cohen class,” IEEE Signal Processing Letters, Vol. 3, No. 2. Feb. 1996.

[5] R. Saxena, and K. Singh, “Fractional Fourier transform: a novel tool for signal processing,” Journal of Indian Inst. Science, 85, pp. 11- 26, 2005.

[6] T. Alieva, and A. Barbe, “Fractional Fourier and radon-wigner transforms of periodic signals,” Elsevier Signal Processing Journal, vol. 69, pp. 183- 189, 1998.

[7] M. A. Kutay, H. M. Ozaktas, O. Arikan, and L. Onural, “Optimal filtering in fractional Fourier domains,” IEEE Trans. On Signal Processing, Vol. 45, No. 5, pp. 1129- 1143, May. 1997.

[8] M. F. Erden, M. A. Kuaty, and H. M. Ozaktas, “Repeated filtering in consecutive fractional Fourier domains and its application to signal restoration,” IEEE Trans. On Signal Processing, Vol. 47, No. 5, pp. 1458- 1462, May 1999.

[9] S. N. Sharma, R. Saxena, and S. C. Saxena, “Tuning of FIR filter transition bandwidth using fractional Fourier transform,” Elsevier Signal Processing Journal, vol. 87, pp. 3147- 3154, 2007.

[10] L. Durak, and S. Aldirmaz, “Adaptive fractional Fourier domain filtering,” Elsevier Signal Processing Journal, Article in Press.

[11] T. Alieva, and M. J. Bastiaans, “On fractional Fourier transform moments,” IEEE Signal Processing Letters, vol. 7, no. 11, pp. 320- 323, November 2000.

[12] L. Stankovic, T. Alieva, M. J. Bastiaans, “Time- frequency analysis based on the windowed fractional Fourier transform,” Elsevier Signal Processing Journal, vol. 83, pp. 2459- 2468, 2003.

[13] L. Cohen, “Time-frequency distributions - a review,” Proceedings of the IEEE, vol. 77, no. 7, pp. 941-980, July 1989.

[14] L. Chen, D. Zhao, “Application of fractional Fourier transform on spatial filtering,” Elsevier Optik Journal, vol. 117, pp. 107- 110, 2006.

[15] S. Shinde, and V. M. Gadre, “An uncertainty principle for real signals in the fractional Fourier transform domain,” IEEE Trans. On Signal Processing, Vol. 49, No. 11, pp. 2545- 2548, Nov. 2001.